UNIVERSITY OF HAWAII
LIBRARY

MAY 9 '50

The PHILOSOPHICAL MAGAZINE

FIRST PUBLISHED IN 1798

. 41 SEVENTH SERIES No. 315 APRIL, 1950

A Journal of Theoretical Experimental and Applied Physics

EDITOR
PROFESSOR N. F. MOTT, M.A., D.Sc., F.R.S.

EDITORIAL BOARD
SIR LAWRENCE BRAGG, O.B.E., M.C., M.A., D.Sc., F.R.S.
ALLAN FERGUSON, M.A., D.Sc.
SIR GEORGE THOMSON, M.A., D.Sc., F.R.S.
PROFESSOR A. M. TYNDALL, C.B.E., D.Sc., F.R.S.

PRICE 10s.
Annual Subscription £5 2s. 6d. payable in advance.

ED AND PUBLISHED BY TAYLOR & FRANCIS LTD., RED LION COURT, FLEET ST., LONDON. E.C.4

Annals of Science

A QUARTERLY REVIEW OF
THE HISTORY OF SCIENCE
SINCE THE RENAISSANCE

EDITORS

D. McKIE, D.Sc., Ph.D.,
University College, London.

HARCOURT BROWN, M.A., Ph.D.,
Brown University, Providence, R.I., U.S.A.

H. W. ROBINSON,
Former Librarian,
Royal Society of London.

ANNUAL SUBSCRIPTION

£2 0s. 0d.

OR

10s. 6d.
PER PART
POST FREE

TAYLOR & FRANCIS, LTD., Red Lion Court, Fleet Street, LONDON, E.C.4

ALERE FLAMMAM

THE MATHEMATICAL WORKS OF JOHN WALLIS, D.D., F.R.S.

by

J. F. SCOTT, Ph.D., B.A.

"His work will be indispensable to those interested in the early history of The Royal Society. I commend to all students of the Seventeenth Century, whether scientific or humane, this learned and lucid book."—Extract from foreword by Prof. E. N. da C. Andrade, D.Sc., Ph.D., F.R.S.

Recommended for publication by University of London

12/6 net

Printed and Published by:

TAYLOR & FRANCIS, LTD.
RED LION COURT, FLEET STREET, LONDON, E.C.4.

XXVI. *Theory of Elasticity in General Coordinates.*

By A. E. Green and W. Zerna,
Department of Mathematics, King's College, Newcastle-upon-Tyne †.

[Received December 22, 1949.]

§ 1. INTRODUCTION.

THE tensor calculus (Ricci-calculus) has for many years been one of the standard tools in certain branches of mathematics and physics as, for example, in the general theory of relativity. So far, however, only a few writers appear to have used tensor calculus for the presentation of a general theory of elasticity, either for infinitesimal or finite displacements. Elasticity theory is often developed in special rectangular cartesian coordinates and, particularly for finite displacements, the algebraic difficulties are considerable. Also, transformation to any other coordinate system is usually difficult and laborious. The tensor treatment avoids these difficulties and is in many ways easier to follow once the elementary ideas of tensor calculus have been mastered.

In 1937 an important paper was published by Murnaghan (1937), in which he obtained some of the basic relations of the theory of elasticity for finite displacements ‡, using general tensor notations. This seems to be a fundamental paper in many respects but the present writers believe that the theory developed there is not yet in its simplest form. Since 1937 papers have been published § in which tensor calculus notations of various types are used for elasticity theory, but the notations and the final forms of some of the results differ considerably, and are not always very simple.

An examination of previous work suggests that this is an appropriate moment in which to attempt a simplification of the presentation of the theory of elasticity in general coordinates. While acknowledging the help derived from previous papers the present writers seek to give a reasonably complete account of elasticity theory in tensor notation, for finite displacements, in which it is believed that the treatment is straightforward, and that the methods used and the relations which are obtained contain many new features.

---

† Communicated by the Authors.

‡ These relations have recently been expressed in vector and dyadic notations by Milne-Thomson (1948).

§ Some of these papers are given in the references.

§ 2. DISPLACEMENT VECTOR. DEFORMATION.

Let every point of a continuous three dimensional body, called briefly the body B_0, be at rest, at time $t=0$, relative to a fixed rectangular Cartesian system of axes x_i whose origin is o. The position vector of a typical point P_0 of the body B_0, referred to o as origin, has the form

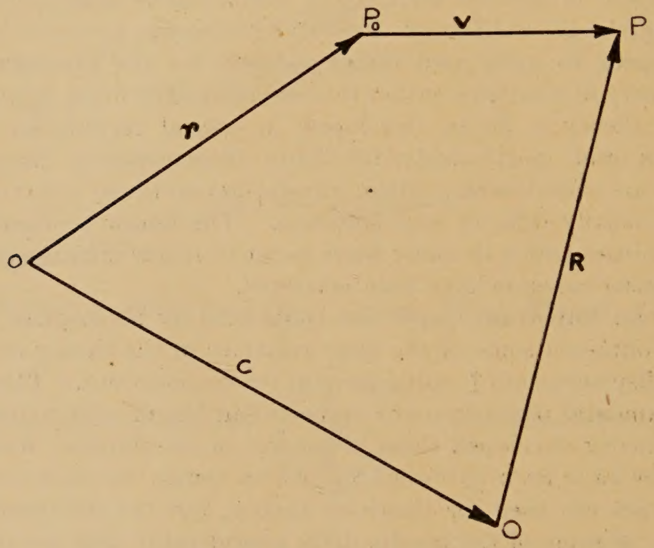
$$\mathbf{r}=\mathbf{r}(x_1, x_2, x_3), \quad \ldots \ldots \ldots \quad (2.1)$$

and the line element is given by

$$ds_0^2=dx^i\,dx^i, \quad \ldots \ldots \ldots \quad (2.2)$$

where the usual summation convention is used and the latin indices take the values 1, 2, 3.

Fig. 1.



We suppose that the positions of the points of the body B_0 change so that at time t a typical point P_0 has moved to P. The point P may be referred to a new fixed rectangular Cartesian system of axes y_i whose origin is at O, and the position vector of P referred to O is (see fig. 1)

$$\mathbf{R}=\mathbf{R}(y_1, y_2, y_3). \quad \ldots \ldots \ldots \quad (2.3)$$

The position vector $\mathbf{v}$ of the point P relative to P_0 represents the displacement of P_0 when the points of B_0 move from their positions at time $t=0$ to their positions at time t. We therefore call

$$\mathbf{v}=\mathbf{R}-\mathbf{r}+\mathbf{c}, \quad \ldots \ldots \ldots \quad (2.4)$$

the displacement vector, where the constant vector $\mathbf{c}$ defines the position of the origin O relative to the origin o.

Except for the assumption that the body B_0 is continuous no other assumptions have been made about the material of the body or about the way in which the motion takes place. We now make some assumptions about the way each point P_0 of B_0 moves to its corresponding position P at time t. This will imply that the material of the body has certain physical properties. We shall not, however, discuss the nature of these physical properties. Although we are guided to some extent by physical ideas in making our assumptions, the justification for applying the theory to physical bodies must rest on the comparison of results obtained from theory with those found by experiment.

We wish to study motions of points of the body B_0 which take place continuously and which are such that any curve in B_0 moves so that it always consists of the same particles. In addition, any line element ds_0 of B_0 always remains of the same order of magnitude. We therefore assume that each point P, at time t, is related to its original position P_0, at time $t=0$, by the equations

$$y_i = y_i(x_1, x_2, x_3, t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.5)$$

where y_i are continuous single-valued functions of x_1, x_2, x_3 and t, and where, for each t, these relations are reversible, so that

$$x_i = x_i(y_1, y_2, y_3, t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

x_i being single-valued functions of y_1, y_2, y_3 for each value of t.

We also assume that, at a given time t, the differentials dy^i exist and are related to the differentials dx^i by the equations

$$dy^i = \frac{\partial y_i}{\partial x_k} dx^k, \qquad dx^i = \frac{\partial x_i}{\partial y_k} dy^k. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.7)$$

These assumptions impose certain restrictions on the partial derivatives of y_i with respect to x_k. Additional restrictions will also be necessary in later parts of the theory but we shall generally suppose, without further discussion, that all conditions which are necessary for the mathematical validity of the theory are satisfied.

If, as a result of the motion of the points of B_0, the relative positions of neighbouring points of the body are altered, so that the length ds_0 of the line element changes to ds, the body B_0 is said to be *strained* or *deformed*. The strained or deformed state of the body, at time t, is denoted by B. The square of the line element in the strained body B may then be written

$$ds^2 = dy^i \, dy^i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.8)$$

where the differentials dy^i are to be evaluated for a fixed time t and are related to dx^i by equation (2.7). If $ds=ds_0$ for all points and for every time t the body is said to be rigid.

The original body B_0 may be described by a general curvilinear set of coordinates θ_i so that

$$x_i = x_i(\theta_1, \theta_2, \theta_3). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.9)$$

We may imagine that this curvilinear coordinate system moves continuously † with the body as we pass from the original state B_0 at time $t=0$ to the state B at time t. It will therefore form a curvilinear system in B so that, from (2.5) and (2.9),

$$y_i = y_i(\theta_1, \theta_2, \theta_3, t). \quad \quad (2.10)$$

Hence, the curvilinear system θ_i may be identified with any convenient coordinate system in either B_0 or B.

We may define in B_0 contravariant components $d\theta^i$ of a vector from the relation (2.9). Thus

$$d\theta^i = \frac{\partial \theta_i}{\partial x_k} dx^k, \qquad dx^k = \frac{\partial x_k}{\partial \theta_i} d\theta^i. \quad \quad (2.11)$$

From (2.7) and (2.11) we have

$$d\theta^i = \frac{\partial \theta_i}{\partial y_k} dy^k, \qquad dy^k = \frac{\partial y_k}{\partial \theta_i} d\theta^i, \quad \quad (2.12)$$

so that $d\theta^i$ remain the contravariant components of a vector under all transformations, including transformations from coordinate systems in B_0 to those in B.

Using (2.9) and (2.10) the position vectors (2.1) and (2.3) of the points P_0 and P respectively take the forms

$$\left.\begin{aligned} \mathbf{r} &= \mathbf{r}(\theta_1, \theta_2, \theta_3), \\ \mathbf{R} &= \mathbf{R}(\theta_1, \theta_2, \theta_3, t), \end{aligned}\right\}, \quad \quad (2.13)$$

and the displacement vector (2.4) becomes

$$\mathbf{v} = \mathbf{v}(\theta_1, \theta_2, \theta_3, t). \quad \quad (2.14)$$

The line elements (2.2) and (2.8) which correspond to the vectors (2.13) are

$$ds_0^2 = g_{ik}\, d\theta^i\, d\theta^k, \quad \quad (2.15)$$

$$ds^2 = G_{ik}\, d\theta^i\, d\theta^k, \quad \quad (2.16)$$

where g_{ik} and G_{ik} are the covariant components of the metric tensors of B_0 and B respectively. The quantities G_{ik} are found for a given time t and are functions of t as well as $\theta_1, \theta_2, \theta_3$.

The covariant components g_{ik} of the metric tensor of B_0, together with the corresponding contravariant components g^{ik}, are derived from the Euclidean space defined by the coordinates x_i. These tensor components satisfy the appropriate laws of transformations for all changes of coordinates of the usual type, including changes to coordinate systems in B. Similarly, the covariant and contravariant components G_{ik}, G^{ik} of the metric tensor of B are derived from the Euclidean space defined by the coordinates y_i, and they satisfy the appropriate laws of transformations for all changes of coordinates, including changes to coordinate systems in B_0.

---

† The values of θ_i which define a point P_0 of B_0 remain fixed with P_0 as it moves from its position in B_0 to its position P in B.

Since both B_0 and B are situated in Euclidean spaces it follows that the components of the Riemann–Christoffel tensors for each body vanish, so that

$$ {}_0R_{ijkl}=0, \qquad R_{ijkl}=0, \quad \ldots\ldots \quad (2.17) $$

the prefix $_0$ referring to the unstrained body B_0. The tensor components ${}_0R_{ijkl}$ are formed from the metric tensor components g_{mn}, g^{mn} and their derivatives with respect to θ_r. Similarly, the tensor components R_{ijkl} are formed from G_{mn} and G^{mn} and their derivatives with respect to θ_r.

§ 3. STRAIN TENSOR.

The state of strain of the body B is characterized † by the difference of the squared line elements (2.15) and (2.16). We write

$$ ds^2-ds_0^2=2\gamma_{ik}\,d\theta^i\,d\theta^k, \quad \ldots\ldots \quad (3.1) $$

where

$$ \gamma_{ik}=\tfrac{1}{2}(G_{ik}-g_{ik}). \quad \ldots\ldots \quad (3.2) $$

The quantities γ_{ik} are the symmetrical covariant components of a tensor which is called the *strain* tensor ‡. When $\gamma_{ik}=0$ for all points, $ds=ds_0$, and the body is rigid.

Since γ_{ik} are symmetrical it is always possible to find a coordinate system in B such that $\gamma_{ik}=0 (i \neq k)$. Since the coordinate system in B is then orthogonal, $G_{ik}=0$ $(i \neq k)$, and hence, from (3.2), $g_{ik}=0$ $(i \neq k)$, so that the original directions of this coordinate system in B_0 are also orthogonal. The directions of the coordinate curves in B which are determined by this coordinate system are called the principal directions of strain; and the strains in the principal directions, the covariant components of the canonical form of the strain tensor, are termed *principal* strains.

With the help of (3.2) we can express the covariant components of the metric tensor of the body B in terms of the covariant components of the metric tensor of the original body B_0 and the covariant components of the strain tensor. Thus

$$ G_{ik}=g_{ik}+2\gamma_{ik} \quad \ldots\ldots \quad (3.3) $$

and we obtain the determinant G of the elements G_{ik} in the form

$$ \begin{aligned} G&=\tfrac{1}{6}e^{ijk}e^{lmn}G_{il}G_{jm}G_{kn} \\ &=g+\tfrac{1}{3}e^{ijk}e^{lmn}\{g_{il}g_{jm}\gamma_{kn}+g_{jm}g_{kn}\gamma_{il}+g_{il}g_{kn}\gamma_{jm} \\ &\quad +2(g_{il}\gamma_{jm}\gamma_{kn}+g_{jm}\gamma_{il}\gamma_{kn}+g_{kn}\gamma_{il}\gamma_{jm})+4\gamma_{il}\gamma_{jm}\gamma_{kn}\}, \quad \ldots \quad (3.4) \end{aligned} $$

---

† The strain may be characterized in other ways as is done, for example, by Richter, H. (1949).

‡ The strain in the neighbourhood of a point can be interpreted in a geometrical way by a strain "quadric", or more specially by a "strain ellipsoid" or "reciprocal strain ellipsoid". See, for example, A. F. H. Love (1927), C. B. Biezeno and K. Grammel (1939).

where g is the determinant of the elements g_{ik} and where e^{ijk} are completely skew-symmetric components of the e-system tensor, which take the values $0, +1, -1$.

In order to calculate the strain components in terms of the displacement vector we write (2.4) in the form

$$\mathbf{R}=\mathbf{r}+\mathbf{v}-\mathbf{c} \tag{3.5}$$

and observe that

$$\mathbf{E}_i=\mathbf{e}_i+\mathbf{v}_{,i}, \tag{3.6}$$

where

$$\mathbf{E}_i=\mathbf{R}_{,i}, \qquad \mathbf{e}_i=\mathbf{r}_{,i}, \tag{3.7}$$

$\mathbf{e}_i$, $\mathbf{E}_i$ being the covariant base vectors in the original and strained bodies respectively. The comma denotes partial differentiation with respect to θ_i. Hence, since

$$g_{ik}=\mathbf{e}_i \cdot \mathbf{e}_k, \qquad G_{ik}=\mathbf{E}_i \cdot \mathbf{E}_k, \tag{3.8}$$

we have, using (3.6),

$$G_{ik}=g_{ik}+\mathbf{e}_i \cdot \mathbf{v}_{,k}+\mathbf{e}_k \cdot \mathbf{v}_{,i}+\mathbf{v}_{,i} \cdot \mathbf{v}_{,k},$$

and therefore

$$\gamma_{ik}=\tfrac{1}{2}(\mathbf{e}_i \cdot \mathbf{v}_{,k}+\mathbf{e}_k \cdot \mathbf{v}_{,i}+\mathbf{v}_{,i} \cdot \mathbf{v}_{,k}). \tag{3.9}$$

The displacement vector may be expressed in terms of the contravariant base vectors $\mathbf{e}^m$ of B_0. Thus

$$\mathbf{v}=v_m \mathbf{e}^m \tag{3.10}$$

and

$$\mathbf{v}_{,i}=v_m|_i \mathbf{e}^m, \tag{3.11}$$

where the vertical line denotes covariant differentiation with respect to B_0 and means

$$v_m|_i=v_{m,i}-\left\{\begin{matrix} r \\ mi \end{matrix}\right\}_0 v_r. \tag{3.12}$$

The Christoffel symbols $\left\{\begin{matrix} r \\ mi \end{matrix}\right\}_0$ in the last expression are to be calculated for B_0 from the components g_{mn}, g^{mn} of the metric tensor of B_0 and are given by

$$\left.\begin{aligned} \left\{\begin{matrix} r \\ mi \end{matrix}\right\}_0 &= g^{rs}[s, mi]_0, \\ [s, mi]_0 &= \tfrac{1}{2}\{g_{si,m}+g_{sm,i}-g_{mi,s}\}. \end{aligned}\right\} \tag{3.13}$$

Introducing (3.10) into (3.9) the strain components (3.9) become

$$\gamma_{ik}=\tfrac{1}{2}(v_i|_k+v_k|_i+v^r|_i v_r|_k), \tag{3.14}$$

if we use the formulæ

$$\mathbf{e}_i \cdot \mathbf{e}^k=\delta_i^k, \tag{3.15}$$

where the Kronecker delta δ_i^k has the usual meaning.

The strain components in (3.14) are expressed in terms of components of the displacement vector with respect to B_0 and their covariant derivatives with respect to B_0. We may, of course, also express the strain-components in terms of components of the displacement vector with respect to B. For this purpose we write

$$\mathbf{r}=\mathbf{R}-\mathbf{v}+\mathbf{c}$$

and we obtain the components of the metric tensor of B_0 in the form

$$g_{ik}=G_{ik}-\mathbf{E}_i\cdot\mathbf{v}_{,k}-\mathbf{E}_k\cdot\mathbf{v}_{,i}+\mathbf{v}_{,i}\cdot\mathbf{v}_{,k}$$

and hence

$$\gamma_{ik}=\tfrac{1}{2}(\mathbf{E}_i\cdot\mathbf{v}_{,k}+\mathbf{E}_k\cdot\mathbf{v}_{,i}-\mathbf{v}_{,i}\cdot\mathbf{v}_{,k}). \quad . \quad . \quad . \quad (3.16)$$

The displacement vector, in terms of the contravariant base vectors $\mathbf{E}^i$ of the body B, is

$$\mathbf{v}=V_m\mathbf{E}^m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.17)$$

and its derivative is

$$\mathbf{v}_{,i}=V_m\|_i\mathbf{E}^m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.18)$$

The double line denotes covariant differentiation with respect to B, that is with respect to θ_i and G_{ik}, so that

$$V_m\|_i=V_{m,i}-\begin{Bmatrix} r \\ mi \end{Bmatrix}V_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.19)$$

where the Christoffel symbols $\begin{Bmatrix} r \\ mi \end{Bmatrix}$ are calculated for B from G_{mn}, G^{mn}. Thus

$$\left.\begin{aligned}\begin{Bmatrix} r \\ mi \end{Bmatrix}&=G^{rs}[s, mi],\\ [s, mi]&=\tfrac{1}{2}\{G_{si,m}+G_{sm,i}-G_{mi,s}\}.\end{aligned}\right\} \quad . \quad . \quad (3.20)$$

Since

$$\mathbf{E}_i\,.\,\mathbf{E}^k=\delta^k_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.21)$$

we find from (3.16) and (3.18) that the strain components now become

$$\gamma_{ik}=\tfrac{1}{2}(V_i\|_k+V_k\|_i-V^r\|_iV_r\|_k), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.22)$$

which gives the strain-components in terms of the components of the displacement with respect to B.

The components of strain in terms of the displacement referred to any convenient set of coordinates in either B_0 or B may at once be written down from (3.14) and (3.22) respectively. For example, the strain-components referred to the cartesian system x_i in B_0 may be found from (3.14) by putting $\theta_i=x_i$, and are

$$\gamma_{11}=\frac{\partial v_1}{\partial x_1}+\frac{1}{2}\left\{\left(\frac{\partial v_1}{\partial x_1}\right)^2+\left(\frac{\partial v_2}{\partial x_1}\right)^2+\left(\frac{\partial v_3}{\partial x_1}\right)^2\right\},$$

$$\gamma_{23}=\frac{1}{2}\left(\frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2}\right)+\frac{1}{2}\left(\frac{\partial v_1}{\partial x_2}\frac{\partial v_1}{\partial x_3}+\frac{\partial v_2}{\partial x_2}\frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2}\frac{\partial v_3}{\partial x_3}\right),$$

etc. Similarly, when at time t the θ_i-curves coincide with the fixed cartesian y_i-curves, so that $\theta_i = y_i$, the strain components referred to y_i may be found from (3.22). Thus

$$\gamma_{11} = \frac{\partial V_1}{\partial y_1} - \frac{1}{2}\left\{\left(\frac{\partial V_1}{\partial y_1}\right)^2 + \left(\frac{\partial V_2}{\partial y_1}\right)^2 + \left(\frac{\partial V_3}{\partial y_1}\right)^2\right\},$$

$$\gamma_{23} = \frac{1}{2}\left(\frac{\partial V_2}{\partial y_3} + \frac{\partial V_3}{\partial y_2}\right) - \frac{1}{2}\left(\frac{\partial V_1}{\partial y_2}\frac{\partial V_1}{\partial y_3} + \frac{\partial V_2}{\partial y_2}\frac{\partial V_2}{\partial y_3} + \frac{\partial V_3}{\partial y_2}\frac{\partial V_3}{\partial y_3}\right),$$

etc.

The mixed components of the strain tensor can be formed in two ways according to the choice of metric. Using the strained metric components G^{ik} we have

$$\epsilon^i_k = G^{il}\gamma_{lk}, \qquad \text{(3.23)}$$

and using the components g^{ik} for the unstrained body,

$$\gamma^i_k = g^{il}\gamma_{lk}. \qquad \text{(3.24)}$$

Similarly, the contravariant components of the strain tensor may be formed in three ways. For our purpose the most convenient definition to take is

$$\epsilon^{ik} = g^{ir}\epsilon^k_r = G^{kl}\gamma^i_l = g^{ir}G^{kl}\gamma_{rl}. \qquad \text{(3.25)}$$

Using (3.2) this becomes

$$\epsilon^{ik} = \tfrac{1}{2}g^{ir}G^{kl}(G_{rl} - g_{rl}), \qquad \text{(3.26)}$$

which reduces to

$$\epsilon^{ik} = \tfrac{1}{2}(g^{ik} - G^{ik}) \qquad \text{(3.27)}$$

or

$$G^{ik} = g^{ik} - 2\epsilon^{ik}, \qquad \text{(3.28)}$$

if we use the results

$$g^{ir}g_{rl} = \delta^i_l, \qquad G^{kl}G_{rl} = \delta^k_l. \qquad \text{(3.29)}$$

§ 4. Equations of Compatibility.

The components γ_{ik} of the strain tensor, as found in the preceding section, are not independent but have to satisfy the condition that the components of the Riemann–Christoffel tensors ${}_0R_{ijkl}$ and R_{ijkl} of the original and strained bodies must vanish. We may write the components of the Riemann–Christoffel tensor of the strained body in the form

$$R_{ijkl} = \tfrac{1}{2}(G_{il,jk} + G_{jk,il} - G_{ik,jl} - G_{jl,ik}) + G^{mn}\{[m, li][n, kj] - [m, ik][n, lj]\}. \qquad \text{(4.1)}$$

Using (3.3) the Cristoffel-symbols $[i, rs]$ of the strained medium become

$$[i, rs] = \tfrac{1}{2}(G_{ir,s} + G_{is,r} - G_{rs,i}) = [i, rs]_0 + 2\gamma_{[i,rs]}, \qquad \text{(4.2)}$$

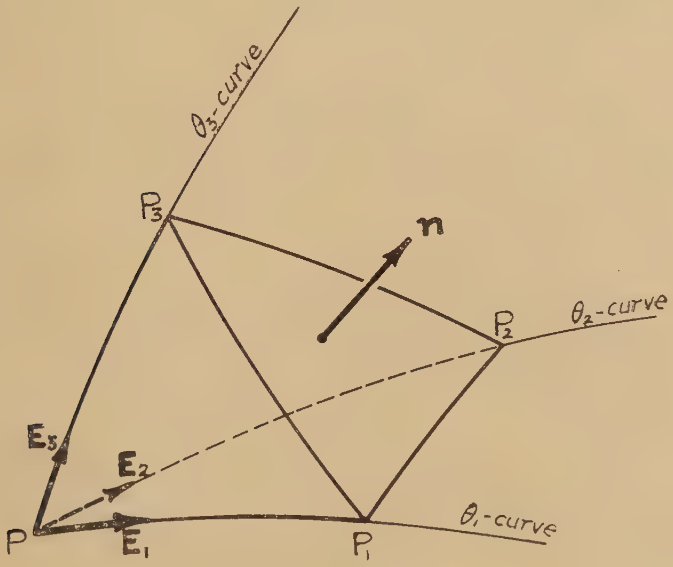
where we have put

$$\gamma_{[i,rs]} = \tfrac{1}{2}(\gamma_{ir,s} + \gamma_{is,r} - \gamma_{rs,i}). \qquad \text{(4.3)}$$

If we now substitute (3.28), (3.3) and (4.2) into (4.1) we obtain

$$\begin{aligned} R_{ijkl} = {}_0R_{ijkl} &+ \gamma_{il,jk} + \gamma_{jk,il} - \gamma_{ik,jl} - \gamma_{jl,ik} \\ &- 2\epsilon^{mn}([m, li]_0[n, kj]_0 - [m, ik]_0[n, lj]_0) \\ &+ 2g^{mn}\{[n, kj]_0\gamma_{[m, li]} + [m, li]_0\gamma_{[n, kj]} \\ &\qquad - [n, lj]_0\gamma_{[m, ik]} - [m, ik]_0\gamma_{[n, lj]}\} \\ &- 4\epsilon^{mn}\{[n, kj]_0\gamma_{[m, li]} + [m, li]_0\gamma_{[n, kj]} - [n, lj]_0\gamma_{[m, ik]} \\ &\qquad - [m, ik]_0\gamma_{[n, lj]} + 2\gamma_{[m, li]}\gamma_{[n, kj]} - 2\gamma_{[m, ik]}\gamma_{[n, lj]}\} \\ &+ 4g^{mn}(\gamma_{[m, li]}\gamma_{[n, kj]} - \gamma_{[m, ik]}\gamma_{[n, lj]}). \qquad (4.4) \end{aligned}$$

Fig. 2.



The conditions

$$ {}_0R_{ijkl} = 0, \quad R_{ijkl} = 0, \qquad (4.5)$$

used in (4.4) give relations between the strain-components which are called the equations of compatibility. As we know that there are only six independent components of the Riemann–Christoffel tensor, (4.4) and (4.5) reduce to six independent equations.

§ 5. EXTERNAL FORCE. STRESS TENSOR.

We assume that the body B_0 is moved to its strained position B by the action of external forces which may vary with the time. At any point P of the strained body B we construct a parallelepiped which is bounded by the faces θ_i=constant, $\theta_i+d\theta^i$=constant, and a tetrahedron whose edges are formed by the coordinate curves PP of lengths ds_i, and the curves P_1P_2, P_2P_3, P_3P_1 (see fig. 2), which in the limit are defined by the three vectors

$$d\mathbf{s}_j - d\mathbf{s}_i = \mathbf{E}_j\, d\theta^j - \mathbf{E}_i\, d\theta^i \qquad (i, j \text{ not summed}). \quad . \quad . \quad . \qquad (5.1)$$

The surfaces θ_i=constant of the tetrahedron have areas

$$\tfrac{1}{2} d\mathrm{S}_i = \tfrac{1}{2}\sqrt{(\mathrm{G}\mathrm{G}^{ii})}\, d\theta^j\, d\theta^k \qquad (i \neq j \neq k,\ i \text{ not summed}), \quad . \qquad (5.2)$$

and the area of $P_1P_2P_3$ is denoted by $\tfrac{1}{2}d\mathrm{S}$.

If $\mathbf{n}$ is a unit vector normal to $P_1P_2P_3$ then

$$\mathbf{n}\, d\mathrm{S} = (d\mathbf{s}_2 - d\mathbf{s}_1) \times (d\mathbf{s}_3 - d\mathbf{s}_1),$$

and, using (5.1), this becomes

$$\left.\begin{aligned} \mathbf{n}\, d\mathrm{S} &= (\mathbf{E}_j \times \mathbf{E}_k)\, d\theta^j\, d\theta^k \\ &= \epsilon_{ijk}\mathbf{E}^i\, d\theta^j\, d\theta^k \end{aligned}\right\} (j, k = 1, 2;\ 2, 3;\ 3, 1), \quad . \quad . \qquad (5.3)$$

where

$$\epsilon_{ijk}\mathbf{E}^i = \mathbf{E}_j \times \mathbf{E}_k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5.4)$$

$$\left.\begin{aligned} \epsilon_{ijk} &= \sqrt{\mathrm{G}} && \text{if } i, j, k \text{ is an even permutation of the numbers } 1, 2, 3, \\ &= -\sqrt{\mathrm{G}} && \text{if } i, j, k \text{ is an odd permutation of the numbers } 1, 2, 3, \\ &= 0 && \text{if any two of the indices } i, j, k \text{ are equal and not summed.} \end{aligned}\right\} \quad . \quad . \quad . \qquad (5.5)$$

Hence from (5.3) and (5.2)

$$\mathbf{n}\, d\mathrm{S} = \sum_i \frac{\mathbf{E}^i}{\sqrt{G^{ii}}}\, d\mathrm{S}_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5.6)$$

If n_i are the covariant components of $\mathbf{n}$, with respect to the base vectors $\mathbf{E}^i$, it follows that †

$$n_i\sqrt{(\mathrm{G}^{ii})}\, d\mathrm{S} = d\mathrm{S}_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5.7)$$

A stress vector $\mathbf{t}$ is associated with the surface $P_1P_2P_3$ of the tetrahedron $PP_1P_2P_3$. The stress associated with the surfaces θ_i=constant of the tetrahedron can be represented in magnitude and direction by quantities $-\mathbf{t}_i$. The quantities $-\mathbf{t}_i$ are not invariant under general transformations of coordinates. This fact is apparent from the notation since vector invariants do not contain suffixes. The suffixes, however, do not indicate that the quantity transforms according to the covariant type of transformation.

By considering the equation of motion of the infinitesimal tetrahedron $PP_1P_2P_3$ we have, in the limit,

$$\mathbf{t}\, d\mathrm{S} = \mathbf{t}_i\, d\mathrm{S}_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5.8)$$

---

† We remember that the summation convention is suspended when an index appears more than twice.

Volume forces and mass-accelerations acting on the tetrahedron do not appear in this equation since they are of a higher order of smallness than the surface forces.

From (5.7) and (5.8) we find that

$$\mathbf{t}=\sum_{i} n_i \mathbf{t}_i \sqrt{G^{ii}}. \qquad (5.9)$$

Since, under general transformations of coordinates, $\mathbf{t}$ is an invariant and n_i are the covariant components of a vector, it follows from (5.9) that $\mathbf{t}_i\sqrt{G^{ii}}$ transforms according to a contravariant type of transformation. We may therefore write

$$\mathbf{t}_i\sqrt{G^{ii}}=\tau^{ik}\mathbf{E}_k=\tau^i_k\mathbf{E}^k, \qquad (5.10)$$

where τ^{ik} and τ^i_k are the contravariant and mixed components of a tensor of the second order which is called the *stress tensor*. Later we will prove that the contravariant components are symmetric so that we are allowed to write the mixed components as τ^i_k and not $\tau^{i}_{\cdot k}$ or $\tau_k^{\cdot i}$.

The stress quantities $\mathbf{t}_i$ exerted by the body in the region $\theta_i>0$ on the body in the region $\theta_i<0$, across the θ_i-surface, has positive values for its components when τ^{ik} are positive.

The covariant components of the stress tensor may be defined by the equations

$$\tau_{ik}=G_{ir}\tau^r_k. \qquad (5.11)$$

From (5.10) we see that the contravariant components τ^{ik} of the stress tensor and the mixed components τ^i_k of the stress tensor are related to the quantities $\mathbf{t}_i\sqrt{G^{ii}}$. The covariant components τ_{ik} of the stress tensor cannot be related to the quantities $\mathbf{t}_i\sqrt{G^{ii}}$ in a simple way and are therefore, in general, of less importance.

From (5.9) and (5.10) we obtain

$$\mathbf{t}=\tau^{ik}n_i\mathbf{E}_k=\tau^i_k n_i\mathbf{E}^k, \qquad (5.12)$$

so that if

$$\mathbf{t}=t^k\mathbf{E}_k=t_k\mathbf{E}^k, \qquad (5.13)$$

then

$$t^k=\tau^{ik}n_i, \qquad t_k=\tau^i_k n_i, \qquad (5.14)$$

and t^k, t_k are the contravariant and covariant components respectively of the stress vector $\mathbf{t}$.

The three stress quantities $\mathbf{t}_i$ in (5.10) may be put in the form

$$\mathbf{t}_i=\sum_k \sqrt{\left(\frac{G_{kk}}{G^{ii}}\right)}\,\tau^{ik}\frac{\mathbf{E}_k}{\sqrt{G_{kk}}}, \qquad (5.15)$$

where $\mathbf{E}_k/\sqrt{G_{kk}}$ are unit vectors along the coordinate curves. The *physical* stress quantities are therefore

$$\sqrt{\left(\frac{G_{kk}}{G^{ii}}\right)}\,\tau^{ik}. \qquad (5.16)$$

These quantities are not, of course, the components of a tensor.

§ 6. EQUATIONS OF MOTION AND EQUILIBRIUM.

We consider the infinitesimal parallelepiped in fig. 3 whose faces have areas dS_i and we consider the equations of motion of all the forces acting on this elementary volume. The forces on the faces θ_i=constant through P are, to the first order, $-\mathbf{t}_i\, dS_i$ (i not summed), and since

$$dS_i = \sqrt{(GG^{ii})}\, d\theta^j\, d\theta^k \qquad (i \text{ not summed, } i \neq j \neq k), \quad . \quad (6.1)$$

we may write this as

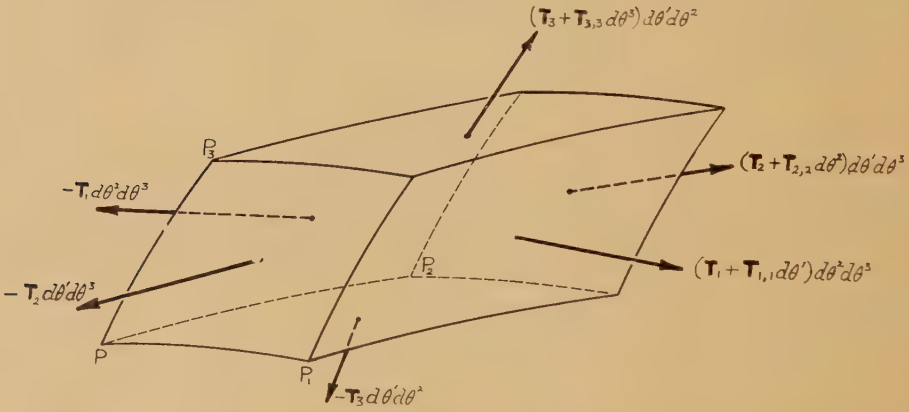
$$-\mathbf{T}_i\, d\theta^j\, d\theta^k \qquad (i \neq j \neq k),$$

where

$$\mathbf{T}_i = \sqrt{(GG^{ii})}\mathbf{t}_i = \sqrt{(G)}\tau^{ik}\mathbf{E}_k = \sqrt{(G)}\tau^i_k\mathbf{E}^k. \quad . \quad . \quad . \quad . \quad (6.2)$$

The forces on the faces θ_i=constant and on the faces $\theta_i + d\theta^i$=constant are shown in fig. 3.

Fig. 3.


$(\mathbf{T}_3+\mathbf{T}_{3,3}d\theta^3)d\theta^1d\theta^2$
P_3
$-\mathbf{T}_1 d\theta^2 d\theta^3$
$(\mathbf{T}_2+\mathbf{T}_{2,2}d\theta^2)d\theta^1d\theta^3$
P_2
$(\mathbf{T}_1+\mathbf{T}_{1,1}d\theta^1)d\theta^2d\theta^3$
$-\mathbf{T}_2 d\theta^1 d\theta^3$
P
P_1
$-\mathbf{T}_3 d\theta^1 d\theta^2$

The body forces acting on the elementary parallelepiped whose volume is $d\tau$ are $\rho\mathbf{F}\, d\tau$ or $\rho\mathbf{Q}\, d\theta^1\, d\theta^2\, d\theta^3$, where

$$\mathbf{Q} = \mathbf{F}\sqrt{G}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

and where ρ is the density of the strained body B.

We now take moments about the centre of gravity of the elementary volume, and, since in the limit we may retain only first order terms, this gives

$$\mathbf{E}_i \times \mathbf{T}_i = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.4)$$

From (6.2), (5.4) and (6.4) we obtain

$$\epsilon_{ikl}\tau^{ik}\mathbf{E}^l = 0.$$

It follows immediately from this equation that

$$\tau^{ik} = \tau^{ki}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.5)$$

so that the contravariant components (and therefore also the covariant components) of the stress tensor are symmetric.

Since the components τ^{ik} are symmetric it is always possible to find a coordinate system in B in which $\tau^{ik}=0$ $(i \neq k)$. The directions of the coordinate curves in such a system are called the principal axes of stress and the stresses which correspond to these axes are called the *principal stresses*.

The remaining equation of motion states that the (vector) sum of all the forces acting on the elementary volume equals the mass-acceleration of the volume, where the acceleration may be taken to be that of the centre of gravity. Hence, in the limit, (see fig. 3)

$$\mathbf{T}_{i,\,i}+\rho\mathbf{Q}=\rho\mathbf{h}, \tag{6.6}$$

where

$$\mathbf{h}=\mathbf{f}\sqrt{G}, \tag{6.7}$$

and $\mathbf{f}$ is the acceleration vector.

When the body is in equilibrium $\mathbf{f}=0$ and

$$\mathbf{T}_{i,\,i}+\rho\mathbf{Q}=0. \tag{6.8}$$

We may express the equation of motion in alternative forms. If

$$\left.\begin{aligned} \mathbf{F}&=F^{i}\mathbf{E}_{i}=F_{i}\mathbf{E}^{i}, \\ \mathbf{f}&=f^{i}\mathbf{E}_{i}=f_{i}\mathbf{E}^{i}, \end{aligned}\right\} \tag{6.9}$$

then

$$\left.\begin{aligned} \mathbf{Q}&=Q^{i}\mathbf{E}_{i}=Q_{i}\mathbf{E}^{i}, \\ \mathbf{h}&=h^{i}\mathbf{E}_{i}=h_{i}\mathbf{E}^{i}, \end{aligned}\right\} \tag{6.10}$$

where

$$\left.\begin{aligned} Q^{i}&=F^{i}\sqrt{G}, \qquad & Q_{i}&=F_{i}\sqrt{G}, \\ h^{i}&=f^{i}\sqrt{G}, \qquad & h_{i}&=f_{i}\sqrt{G}. \end{aligned}\right\} \tag{6.11}$$

Also, if

$$\sigma^{ik}=\tau^{ik}\sqrt{G}, \qquad \sigma^{i}_{k}=\tau^{i}_{k}\sqrt{G}, \tag{6.12}$$

then, from (6.2),

$$\mathbf{T}_{i}=\sigma^{ik}\mathbf{E}_{k}=\sigma^{i}_{k}\mathbf{E}^{k}. \tag{6.13}$$

It should be noticed that the quantities defined in (6.11) and (6.12) are not components of tensors†. Using (6.9) to (6.13) the equation of motion (6.6) becomes

$$\sigma^{ik}{}_{,i}+\begin{Bmatrix} k \\ ir \end{Bmatrix}\sigma^{ir}+\rho Q^{k}=\rho h^{k}, \tag{6.14}$$

or

$$\sigma^{i}_{k,\,i}-\begin{Bmatrix} r \\ ki \end{Bmatrix}\sigma^{i}_{r}+\rho Q_{k}=\rho h_{k}. \tag{6.15}$$

Since

$$\begin{Bmatrix} i \\ ik \end{Bmatrix}=\frac{1}{\sqrt{G}}\,\frac{\partial\sqrt{G}}{\partial\theta_{k}},$$

---

† They are sometimes called relative or weighted tensors.

these equations may be reduced respectively to

$$\tau^{ik}{}_{,i}+\left\{\begin{matrix}k\\ ir\end{matrix}\right\}\tau^{ir}+\left\{\begin{matrix}r\\ ir\end{matrix}\right\}\tau^{ik}+\rho \mathrm{F}^{k}=\rho f^{k}, \quad . \quad . \quad . \quad . \quad (6.16)$$

$$\tau^{i}_{k,i}-\left\{\begin{matrix}r\\ ki\end{matrix}\right\}\tau^{i}_{r}+\left\{\begin{matrix}r\\ ir\end{matrix}\right\}\tau^{i}_{k}+\rho \mathrm{F}_{k}=\rho f_{k}. \quad . \quad . \quad . \quad . \quad (6.17)$$

These may be written more compactly in the tensor forms

$$\tau^{ik}\|_{i}+\rho \mathrm{F}^{k}=\rho f^{k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.18)$$

$$\tau^{i}_{k}\|_{i}+\rho \mathrm{F}_{k}=\rho f_{k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.19)$$

where the double line again denotes covariant differentiation with respect to the strained body B.

With the help of (5.11) it is possible, of course, to write the equations of motion in terms of the covariant components of the stress tensor but this form appears to be of little interest and will not be given here.

The equations (6.6) and (6.14) to (6.19) are alternative forms of the equations of motion. The most convenient form to use depends on the type of problem which is being examined but the equations (6.6), (6.18) and (6.19) are particularly simple in form.

Finally, the conditions at the boundary surface of the body at which the surface forces are prescribed require that,

$$\mathbf{t}=\mathbf{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.20)$$

where **P** is the surface force vector. If we express the surface forces **P** in terms of their components,

$$\mathbf{P}=\mathrm{P}^{k}\mathbf{E}_{k}=\mathrm{P}_{k}\mathbf{E}^{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.21)$$

and use (5.12), the boundary conditions have the alternative forms

$$\tau^{ik}n_{i}=\mathrm{P}^{k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.22)$$

or

$$\tau^{i}_{k}n_{i}=\mathrm{P}_{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.23)$$

§ 7. VIRTUAL WORK.

When the body is in the strained state B at time t the displacement **v** of each point P which was originally at P_0 is

$$\mathbf{v}=\mathbf{v}(\theta_1, \theta_2, \theta_3, t).$$

We now consider any other geometrically possible motion which is such that, during the same time t, the points P_0 move to P′ and form a strained body B′, such that the displacement of P_0 is of the form

$$\mathbf{v}'=\mathbf{v}(\theta_1, \theta_2, \theta_3, t)+\epsilon\mathbf{w}(\theta_1, \theta_2, \theta_3, t),$$

where $\mathbf{w}$ is an arbitrary vector and ϵ is a real number which we shall take to be small, so that its squares and higher powers can be neglected compared with ϵ. We call $\epsilon\mathbf{w}$ a variation or virtual displacement of $\mathbf{v}$ and write

$$\delta\mathbf{v}=\mathbf{v}'-\mathbf{v}=\epsilon\mathbf{w}.$$

This corresponds to the usual definition of variation in the calculus of variations[†].

The variation of $\mathbf{v}_{,i}$ is defined to be

$$\delta(\mathbf{v}_{,i})=\mathbf{v}'_{,i}-\mathbf{v}_{,i}=\epsilon\mathbf{w}_{,i}=(\epsilon\mathbf{w})_{,i}=(\delta\mathbf{v})_{,i}, \quad . \quad . \quad . \quad . \quad (7.1)$$

so that δ and partial derivatives with respect to θ_i are interchangeable.

The variations of the covariant base vectors $\mathbf{E}_i$ are given by

$$\delta\mathbf{E}_i=\delta(\mathbf{e}_i+\mathbf{v}_{,i})=(\delta\mathbf{v})_{,i},$$

if we remember that $\mathbf{e}_i$ does not depend on $\mathbf{v}$. The variations of the scalar-products of the base vectors are found to be

$$\delta(\mathbf{E}_i\,.\,\mathbf{E}_k)=\mathbf{E}_i\,.\,\delta\mathbf{E}_k+\mathbf{E}_k\,.\,\delta\mathbf{E}_i.$$

Hence, using (7.1) and (3.2), the variation of the components of the strain tensor are

$$2\delta\gamma_{ik}=\delta G_{ik}=\mathbf{E}_i\,.\,(\delta\mathbf{v})_{,k}+\mathbf{E}_k\,.\,(\delta\mathbf{v})_{,i}, \quad . \quad . \quad . \quad . \quad (7.2)$$

since $\delta g_{ik}=0$.

We now consider the whole strained body and calculate the work done, during a virtual displacement, by the body forces and the surface forces acting on its boundary. This is called the virtual work and is defined[‡] to be

$$\delta^*A=\int_S \mathbf{P}\,.\,\delta\mathbf{v}\,dS+\int_\tau \rho\mathbf{F}\,.\,\delta\mathbf{v}\,d\tau, \quad . \quad . \quad . \quad . \quad (7.3)$$

where dS is an element of the bounding surface and $d\tau$ a volume element. In (7.3) the surface integral is to be taken over the whole of the bounding surface S. The volume integral is throughout the volume of B. In order to be able to evaluate the surface integral in (7.3) we assume that the surface forces are prescribed over parts of the surface and that the surface displacements are prescribed over the remainder of the surface. If the variation of $\mathbf{v}$ is then chosen to be zero where the surface displacements are prescribed, the surface integral is zero over these parts and may be evaluated over the remainder of the surface over which the surface forces are known.

Using the boundary conditions (6.20) the expression for δ^*A may be written

$$\delta^*A=\int_S \mathbf{t}\,.\,\delta\mathbf{v}\,dS+\int_\tau \rho\mathbf{F}\,.\,\delta\mathbf{v}\,d\tau. \quad . \quad . \quad . \quad . \quad (7.4)$$

[†] See, for example, Courant and Hilbert (1931).

[‡] The star on δ^* indicates that δ^*A is not the variation of a function A.

From (5.12) and (6.2)

$$\mathbf{t}=\frac{n_i}{\sqrt{\mathrm{G}}}\mathbf{T}_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.5)$$

so that (7.4) becomes

$$\delta^*\mathrm{A}=\int_{\mathrm{S}}\frac{n_i}{\sqrt{\mathrm{G}}}\mathbf{T}_i\,.\,\delta\mathbf{v}\,d\mathrm{S}+\int_\tau\rho\mathbf{F}\,.\,\delta\mathbf{v}\,d\tau.$$

The surface integral may be transformed by using Green's theorem in the form

$$\int_{\mathrm{S}}\mathrm{B}^r n_r\,d\mathrm{S}=\int_\tau\frac{\{\mathrm{B}\sqrt{\mathrm{G}}\}_{,r}}{\sqrt{\mathrm{G}}}d\tau, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.6)$$

where B^r are the contravariant components of a vector. Thus

$$\delta^*\mathrm{A}=\int_\tau\frac{(\mathbf{T}_i\,.\,\delta\mathbf{v})_{,i}}{\sqrt{\mathrm{G}}}d\tau+\int_\tau\rho\mathbf{F}\,.\,\delta\mathbf{v}\,d\tau=\int_\tau\{(\mathbf{T}_{i,i}+\rho\mathbf{Q})\delta\mathbf{v}+\mathbf{T}_i\,.\,(\delta\mathbf{v})_{,i}\}\frac{d\tau}{\sqrt{\mathrm{G}}},$$

and using (6.2) and (6.6) this reduces to

$$\delta^*\mathrm{A}=\int_\tau\{\rho\mathbf{f}\,.\,\delta\mathbf{v}+\tau^{ik}\mathbf{E}_k\,.\,(\delta\mathbf{v})_{,i}\}\,d\tau. \quad . \quad . \quad . \quad . \quad . \quad (7.7)$$

Since τ^{ik} are symmetric components of a tensor

$$\delta^*\mathrm{A}=\int_\tau\{\rho\mathbf{f}\,.\,\delta\mathbf{v}+\tfrac{1}{2}\tau^{ik}[\mathbf{E}_k\,.\,(\delta\mathbf{v})_{,i}+\mathbf{E}_i\,.\,(\delta\mathbf{v})_{,k}]\}\,d\tau$$

$$=\iiint\{\rho\mathbf{f}\,.\,\delta\mathbf{v}+\tau^{ik}\delta\gamma_{ik}\}\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3, \quad . \quad . \quad . \quad (7.8)$$

if we use (7.2).

Subject to boundary conditions at the surface of the body, the virtual displacement is an arbitrary geometrically possible displacement, so that equation (7.8), as is seen from its derivation, can replace the equations of motion of the body.

When the body is in equilibrium, $\mathbf{f}=0$, and (7.8) reduces to †

$$\delta^*\mathrm{A}=\iiint\tau^{ik}(\delta\gamma_{ik})\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3. \quad . \quad . \quad . \quad . \quad . \quad (7.9)$$

This represents the necessary (and sufficient) conditions that the body is in equilibrium and may replace the equations of equilibrium.

For a rigid virtual displacement $\delta\gamma_{ik}=0$ and therefore

$$\delta^*\mathrm{A}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.10)$$

This states that for a body in equilibrium the virtual work of the external forces is zero for any rigid virtual displacements.

---

† This result is a general form of the well-known principle of virtual work of elastic bodies.

§ 8. Strain-Energy Function.

We now confine our attention to a body B_0 which at time $t=0$ is unstrained and unstressed †, and we assume that B is obtained from B_0 entirely by the action of external forces ‡. If B is able to recover its original size and shape *completely* when the external forces cease to act the body is said to be elastic. This implies that when all the components of the strain tensor vanish then all the components of the stress tensor vanish and vice versa.

An element of mass of the strained body B at time t is given by

$$dm=\rho\sqrt{G}\,d\theta^1\,d\theta^2\,d\theta^3, \qquad (8.1)$$

and the principle of conservation of mass in a virtual displacement is expressed by

$$\delta(dm)=0. \qquad (8.2)$$

We now consider the actual motion of the body B which takes place in time dt so that the displacement vector increases by $d\mathbf{v}=\frac{\partial \mathbf{v}}{\partial t}$, θ_i being fixed. Let dK denote the increment of kinetic energy of B during the time dt. The kinetic energy K is given by

$$\mathrm{K}=\tfrac{1}{2}\int_{\mathrm{B}}\left(\frac{\partial \mathbf{v}}{\partial t}\right)^2 dm, \qquad (8.3)$$

so that

$$d\mathrm{K}=\int\frac{\partial \mathbf{v}}{\partial t}\,.\,d\left(\frac{\partial \mathbf{v}}{\partial t}\right)dm,$$

since the mass element dm is unaltered. But $d\left(\frac{\partial \mathbf{v}}{\partial t}\right)$ is the increase of velocity in time dt and therefore

$$d\left(\frac{\partial \mathbf{v}}{\partial t}\right)=\frac{\partial^2 \mathbf{v}}{\partial t^2}\,dt=\mathbf{f}\,dt.$$

Hence

$$d\mathrm{K}=\int_{\mathrm{B}}\mathbf{f}\,.\,d\mathbf{v}\,dm=\iiint(\mathbf{f}\,.\,d\mathbf{v})\rho\sqrt{G}\,d\theta^1\,d\theta^2\,d\theta^3. \qquad (8.4)$$

The actual increase in the work done d^*A by the external forces acting on B, during time dt, may be obtained by putting $\delta\mathbf{v}=d\mathbf{v}$ and

$$\delta\gamma_{ik}=d\gamma_{ik}=\frac{\partial \gamma_{ik}}{\partial t}\,dt,$$

---

† This means, for example, that no stresses are induced in B_0 by its manufacture or by the action of body forces.

‡ Strains produced without external forces but by changes of temperature are not considered.

in (7.8), where $d\gamma_{ik}$ are the increases in the components of strain in time dt. Thus

$$d^*\mathrm{A}=\iiint(\rho\mathbf{f}\,.\,d\mathbf{v}+\tau^{ik}\,d\gamma_{ik})\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3. \quad . \quad . \quad . \quad (8.5)$$

The fundamental energy-conservation laws of thermodynamics for reversible processes may be written†

$$\int_{\mathrm{B}}\mathrm{T}\,d\sigma\,dm+d^*\mathrm{A}=\int_{\mathrm{B}}d\mathrm{U}\,dm+d\mathrm{K}, \quad . \quad . \quad . \quad . \quad (8.6)$$

where T is the temperature of the mass element dm, $d\sigma$ the increment of entropy per unit mass and dU the increment of internal energy per unit mass. By using (8.4) and (8.5) this equation reduces to

$$\iiint(d\mathrm{U}-\mathrm{T}\,d\sigma)\rho\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3=\iiint\tau^{ik}(d\gamma_{ik})\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3. \quad (8.7)$$

Since we may consider any arbitrary volume, it follows that

$$(d\mathrm{U}-\mathrm{T}\,d\sigma)\rho=\tau^{ik}\,d\gamma_{ik}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.8)$$

if we make suitable assumptions about the continuity of the integrands.

If the changes which have taken place in time dt are reversible and isothermal, then T is constant, and

$$d\mathrm{W}=d(\mathrm{U}-\mathrm{T}\sigma)=\frac{\tau^{ik}}{\rho}\,d\gamma_{ik}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.9)$$

where dW is a complete differential. The quantities $d\gamma_{ik}$ are also, for fixed values of θ_i, complete differentials. Since we are assuming that the body is elastic, the changes which take place are reversible, and the condition of the body B is independent of the actual path taken to reach the state B from B_0. The differentials $d\gamma_{ik}$ in (8.9) may therefore be considered as arbitrary so that, since dW is a complete differential,

$$\tau^{ik}=\rho\frac{\partial\mathrm{W}}{\partial\gamma_{ik}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.10)$$

where W is a function of γ_{ik} and all changes of state are isothermal.

Similarly, if the changes in an elastic body are adiabatic, so that $d\sigma=0$, we find from (8.8)

$$\tau^{ik}=\rho\frac{\partial\mathrm{U}}{\partial\gamma_{ik}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.11)$$

where U is a function of γ_{ik}.

The function W and U are called strain-energy functions and play the rôle of potential energy per unit mass for the isothermal and adiabatic cases respectively.

---

† Under these conditions dU and $d\sigma$ are complete differentials.

The relations (8.10) and (8.11) between the contravariant components of the stress tensor and the strain-energy function is the most convenient for our purpose, but it is of interest to derive an alternative form†. The mixed components ϵ^i_k of the strain tensor are defined in (3.23) and from this definition

$$\gamma_{kl}=\epsilon^i_l G_{ki}=\epsilon^i_l(g_{ki}+2\gamma_{ki})$$

Hence

$$d\gamma_{kl}=G_{ki}\,d\epsilon^i_l+2\epsilon^i_l\,d\gamma_{ki}$$

and therefore

$$G^{mk}(\delta^i_l-2\epsilon^i_l)d\gamma_{ki}=G^{mk}G_{ki}\,d\epsilon^i_l=d\epsilon^m_l. \quad . \quad . \quad . \quad . \quad (8.12)$$

From (8.10)

$$\rho\,dW=\tau^{ik}\,d\gamma_{ik}=G^{mk}\tau^i_m\,d\gamma_{ki},$$

and with (8.12) this becomes

$$\rho(\delta^i_l-2\epsilon^i_l)\,dW=\tau^i_m\,d\epsilon^m_l,$$

so that ‡, for the reversible isothermal case,

$$\tau^i_m=\rho\left(\frac{\partial W}{\partial \epsilon^m_i}-2\epsilon^i_l\frac{\partial W}{\partial \epsilon^m_l}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.13)$$

with a similar result with U written for W when the changes are reversible and adiabatic.

§ 9. VARIATIONAL, WORK AND ENERGY PRINCIPLES.

In this, and in subsequent sections, we shall consider an elastic body which undergoes isothermal changes of state and use the strain-energy function W. The corresponding results for adiabatic changes may be obtained in a similar manner.

If W′ is the total strain-energy of the whole body, then

$$W'=\iiint \rho W\sqrt{G}\,d\theta^1\,d\theta^2\,d\theta^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.1)$$

and, remembering § (8.2),

$$\delta W'=\iiint (\delta W)\rho\sqrt{G}\,d\theta^1\,d\theta^2\,d\theta^3$$

$$=\iiint \frac{\partial W}{\partial \gamma_{ik}}(\delta\gamma_{ik})\rho\sqrt{G}\,d\theta^1\,d\theta^2\,d\theta^3. \quad . \quad . \quad . \quad (9.2)$$

---

† Other relations between the components of the stress tensor and the strain energy may be found by using the expression (7.7) for δ^*A instead of (7.8).

‡ If our coordinates θ_i are identified with a general coordinate system in the strained body B at a given time *t*, then (8.13) agrees with a similar result obtained by a somewhat different argument by F. D. Murnaghan (1937). On the other hand, if we take θ_i to be a general coordinate system in the original body B_0 the result (8.10) is the same as that given by Murnaghan when his two coordinate systems coincide.

§ The δ used here is the usual variation symbol of the Calculus of Variations.

Hence, from (7.8), (8.10) and (9.2), we have

$$\delta^*\mathrm{A}=\delta\mathrm{W}'+\iiint(\mathbf{f}\,.\,\delta\mathbf{v})\rho\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3. \quad . \quad . \quad . \quad (9.3)$$

We integrate this last result between two arbitary times t_0 and t_1. Thus

$$\int_{t_0}^{t_1}\delta^*\mathrm{A}\,dt=\int_{t_0}^{t_1}\delta\mathrm{W}'\,dt+\int_{t_0}^{t_1}\iiint(\mathbf{f}\,.\,\delta\mathbf{v})\rho\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3\,dt. \quad . \quad (9.4)$$

The last integral may be transformed by partial integration. Using $\mathbf{f}=\dfrac{\partial^2\mathbf{v}}{\partial t^2}$, and remembering that the mass element $\rho\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3$ is constant, we obtain

$$\int_{t_0}^{t_1}\iiint(\mathbf{f}\,.\,\delta\mathbf{v})\rho\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3\,dt=\left[\iiint\left(\frac{\partial\mathbf{v}}{\partial t}\,.\,\delta\mathbf{v}\right)\rho\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3\right]_{t_0}^{t_1}$$
$$-\int_{t_0}^{t_1}\iiint\frac{\partial\mathbf{v}}{\partial t}\,.\,\frac{\partial\delta\mathbf{v}}{\partial t}\,\rho\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3\,dt.$$

We now suppose that the variation $\delta\mathbf{v}$ is such that it vanishes at time t_0 and t_1 so that

$$\int_{t_0}^{t_1}\iiint(\mathbf{f}\,.\,\delta\mathbf{v})\rho\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3\,dt=-\int_{t_0}^{t_1}\delta\mathrm{K}\,dt, \quad . \quad . \quad (9.5)$$

where K, the kinetic energy of the strained body B, is given by (8.3), and

$$\delta\mathrm{K}=\iiint\frac{\partial\mathbf{v}}{\partial t}\,.\,\delta\left(\frac{\partial\mathbf{v}}{\partial t}\right)\rho\sqrt{\mathrm{G}}\,d\theta^1\,d\theta^2\,d\theta^3. \quad . \quad . \quad . \quad . \quad (9.6)$$

Equation (9.4) now becomes

$$\int_{t_0}^{t_1}\delta^*\mathrm{A}\,dt=\int_{t_0}^{t_1}\delta\mathrm{W}'\,dt-\int_{t_0}^{t_1}\delta\mathrm{K}\,dt. \quad . \quad . \quad . \quad . \quad (9.7)$$

If the external forces can be derived from a potential energy function Ω then

$$\delta^*\mathrm{A}=-\delta\Omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.8)$$

and (9.7) reduces to the simple form

$$\int_{t_0}^{t_1}\delta(\mathrm{K}-\mathrm{W}'-\Omega)dt=\delta\int_{t_0}^{t_1}(\mathrm{K}-\mathrm{W}'-\Omega)dt=0, \quad . \quad . \quad . \quad (9.9)$$

where $\mathrm{K}-\mathrm{W}'-\Omega$ is the Lagrangian function.

Equation (9.9) is the analytical expression of Hamilton's Principle, which says that the variation of the integral of the Lagrangian function, between times t_0 and t_1, takes a stationary value, provided that the variation of the displacement vector is taken in such a way that it vanishes at the times t_0, t_1.

If the strained body B is in equilibrium then $\mathbf{f}=0$ and the virtual work (9.3) of the external forces acting on the body becomes

$$\delta^*\mathrm{A}=\delta\mathrm{W}',$$

and, if (9.8) still holds,

$$\delta(\mathrm{W}'+\Omega)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.10)$$

This states that the variation of the total potential energy $W'+\Omega$ has a stationary value.

The principles (9.9) and (9.10) were found from the expression (9.3) for the virtual work of the external forces. We may also derive from (9.3) expressions for the *actual* work performed in time dt by replacing $\delta\mathbf{v}$ by the actual displacement $d\mathbf{v}$. The virtual work of the external forces δ^*A now becomes the actual increase d^*A of work performed in time dt, and $\delta W'$ becomes dW', and therefore (9.3) is now

$$d^*A=dW'+\iiint(\mathbf{f}\,.\,d\mathbf{v})\rho\sqrt{G}\,d\theta^1\,d\theta^2\,d\theta^3. \quad . \quad . \quad . \quad (9.11)$$

If we use (8.4) this reduces to

$$d^*A=dW'+dK. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.12)$$

This states that the work done by the external forces in time dt is equal to the sum of the increase of the kinetic energy and the increase of the strain-energy.

§ 10. STRESS-STRAIN RELATIONS.

We have seen in § 8 that, under certain conditions, the stresses which occur in an elastic body are related to the function W by the equations

$$\tau^{ik}=\rho\frac{\partial W}{\partial\gamma_{ik}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.1)$$

and

$$W'=\iiint\rho W\sqrt{G}\,d\theta^1\,d\theta^2\,d\theta^3, \quad . \quad . \quad . \quad . \quad . \quad (10.2)$$

represents the total strain-energy of the strained body. Since mass elements are conserved under straining, we have

$$\rho_0\sqrt{g}=\rho\sqrt{G}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.3)$$

where ρ_0 is the density of the unstrained body. Hence equations (10.1) may be written

$$\sqrt{(G)}\tau^{ik}=\rho_0\sqrt{(g)}\frac{\partial W}{\partial\gamma_{ik}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.4)$$

At this point it is necessary to make further assumptions if we wish to obtain an explicit relation between stress and strain. The validity of the assumptions when applied to physical bodies must be tested by referring to experiment. The relation between stress and strain, in the theory which has been developed so far, is of such a form that a strain-energy function W exists with the property (10.4). We therefore assume a suitable form for W and derive the stress-strain relations from (10.4).

Since the density ρ_0 of the unstrained body, and W, are scalars, the quantity $\rho_0 W$ is invariant under all transformations of coordinates and we assume that W can be expanded as a power series † in the strain components γ_{ik}. Thus

$$\rho_0 W = E + E^{ik}\gamma_{ik} + F^{iklr}\gamma_{ik}\gamma_{lr} + F^{iklrmn}\gamma_{ik}\gamma_{lr}\gamma_{mn} + \dots, \qquad (10.5)$$

The functions E^{ik}, F^{iklr}, . . . do not depend on γ_{ik} and since W is invariant they must be components of tensors. If W is taken to be zero when the body is in the initial state B in which $\gamma_{ik}=0$ then $E=0$.

From (10.4) and (10.5) we have ‡

$$\sqrt{(G/g)}\,\tau^{ik} = E^{ik} + E^{iklr}\gamma_{lr} + E^{iklrmn}\gamma_{lr}\gamma_{mn} + \dots, \qquad (10.6)$$

where we have put

$$\left.\begin{aligned} E^{iklr} &= F^{iklr} + F^{lrik}, \\ E^{iklrmn} &= F^{iklrmn} + F^{lrikmn} + F^{mnlrik}. \end{aligned}\right\} \qquad (10.7)$$

Since we have assumed that our material is elastic there is no stress when the strains are zero provided there is no initial strain. Hence

$$E^{ik}=0.$$

The expression (10.7) gives us a general relation between stress and strain, and we call the coefficients E^{iklr}, E^{iklrmn}, . . . the elastic coefficients of the body. They depend on the properties of the body in the unstrained state.

The quantities E^{iklr} are coefficients in an expansion of the strain-energy function W in terms of the components of strain and therefore refer to an elastic body which undergoes isothermal changes of state. When the changes are adiabatic the strain-energy function is U and the corresponding elastic coefficients are, in general, different, but for many purposes these differences are not great and can be ignored.

From the symmetric properties of the components of the stress and strain tensors, and the relations (10.7), we deduce the following symmetric properties :

$$E^{iklr} = E^{kilr} = E^{ikrl} = E^{lrik}, \qquad (10.8)$$

with similar properties for E^{iklrmn}, ...

Because of the symmetric properties of the elastic coefficients, we see, from (10.4) and (10.6), that the strain-energy (10.5) may be expressed in the more convenient form

$$\rho_0 W = \tfrac{1}{2}E^{iklr}\gamma_{ik}\gamma_{lr} + \tfrac{1}{3}E^{iklrmn}\gamma_{ik}\gamma_{lr}\gamma_{mn}. \qquad (10.9)$$

In order to examine the properties of the elastic coefficients in more detail we refer all quantities to the rectangular coordinates x_i in the unstrained body B_0 so that $\theta_i = x_i$ and $g=1$. In these coordinates the tensor components E^{iklr}, E^{iklrmn}, . . . are denoted by c^{iklr}, c^{iklrmn}, and the covariant components γ_{mn} of the strain tensor by e_{mn}. The contravariant components of the stress tensor are denoted by t^{ik} and these are referred to coordinates in the strained body B which, in the unstrained body B_0, were the coordinates x_i. The strain-energy function then becomes

---

† This, of course, is not the only possible assumption that can be made.

‡ In using (10.4) we suppose that γ_{ik} is distinct from γ_{ki}.

$$\rho_0 W = \tfrac{1}{2}c^{iklr}e_{ik}e_{lr} + \tfrac{1}{3}c^{iklrmn}e_{ik}e_{lr}e_{mn} + \ldots \quad . \ . \ . \quad (10.10)$$

and

$$t^{ik}\sqrt{(G)} = c^{iklr}e_{lr} + c^{iklrmn}e_{lr}e_{mn} + \ldots . \quad . \ . \ . \quad (10.11)$$

If the elastic coefficients c^{iklr}, c^{iklrmn}, . . . are independent of the orientation of the rectangular axes x_i at each point we say that the body is *elastically isotropic*, or more simply, *isotropic*. Otherwise the body is said to be æolotropic or anisotropic.

If the density of the original body B_0 is constant and if the elastic coefficients c^{iklr}, c^{iklrmn}, . . . are constants, we say that the body is *elastically homogeneous*. In this case the covariant derivatives of the general elastic coefficients E^{iklr}, E^{iklrmn}, . . . are zero, so that

$$E^{iklr}{}_{|t} = 0, \qquad E^{iklrmn}{}_{|t} = 0. \quad . \ . \ . \ . \ . \ . \ . \quad (10.12)$$

In the remainder of this paper we shall restrict our attention to bodies which are homogeneous in the above sense so that equations (10.12) hold. Similar results, however, hold for non-homogeneous bodies. Because of theoretical and experimental difficulties we restrict our attention to those materials for which the coefficients E^{iklrmn}, . . . are sufficiently small, so that we can express the strain-energy W approximately in the form

$$\rho_0 W = \tfrac{1}{2}E^{iklr}\gamma_{ik}\gamma_{lr} = \tfrac{1}{2}c^{iklr}e_{ik}e_{lr}, \quad . \ . \ . \ . \quad (10.13)$$

with the corresponding stress-strain relations

$$\sqrt{(G)}\tau^{ik} = \sqrt{(g)}E^{iklr}\gamma_{lr}, \quad . \ . \ . \ . \ . \quad (10.14)$$

$$\sqrt{(G)}t^{ik} = c^{iklr}e_{lr}. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (10.15)$$

The strain energy in this case can be also put in the alternative form

$$\rho W = \tfrac{1}{2}\tau^{ik}\gamma_{ik}. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (10.16)$$

§ 11. ELASTIC COEFFICIENTS.

In general, remembering the symmetric properties

$$c^{iklr} = c^{kilr} = c^{ikrl} = c^{lrik}, \quad . \ . \ . \ . \ . \ . \ . \quad (11.1)$$

we have 21 elastic coefficients c^{iklr} for an æolotropic body. If the coordinate system x_i is changed to another rectangular cartesian system z_i then the coefficients c^{iklr} become d^{iklr} and, in view of their tensor character,

$$d^{iklr} = \frac{\partial z_i}{\partial x_m}\frac{\partial z_k}{\partial x_n}\frac{\partial z_l}{\partial x_s}\frac{\partial z_r}{\partial x_t}c^{mnst}. \quad . \ . \ . \ . \ . \ . \quad (11.2)$$

Symmetry of structure of the general æolotropic body will introduce relations among the elastic coefficients c^{iklr} which may be found directly by using (11.2) and suitable transformations of coordinates, but we will not examine these in detail here. When the body is homogeneous and isotropic the coefficients c^{iklr} are invariant under all transformations to other rectangular cartesian coordinates and it can be shown that the coefficients reduce to

$$c^{mnst}=\lambda\delta^{mn}\delta^{st}+\mu\delta^{ms}\delta^{nt}+\mu\delta^{mt}\delta^{ns}, \quad . \quad . \quad . \quad . \quad (11.3)$$

where λ, μ are constants. Hence †

$$\mathrm{E}^{iklr}=\lambda g^{ik}g^{lr}+\mu(g^{il}g^{kr}+g^{ir}g^{kl}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.4)$$

or

$$\mathrm{E}^{iklr}=\mu\left\{g^{il}g^{kr}+g^{ir}g^{kl}+\frac{2\eta}{1-2\eta}\,g^{ik}g^{lr}\right\}. \quad . \quad . \quad . \quad (11.5)$$

where

$$\lambda=\frac{2\eta\mu}{1-2\eta}.$$

From (10.14) and (11.5) we find that the stress-strain relation for an isotropic body takes the form

$$\surd(\mathrm{G})\tau^{ik}=\mu\surd(g)\left\{g^{il}g^{kr}+g^{ir}g^{kl}+\frac{2\eta}{1-2\eta}\,g^{ik}g^{lr}\right\}\gamma_{lr}, \quad . \quad . \quad (11.6)$$

or

$$\surd(\mathrm{G})\tau^{ik}=\mu\surd(g)\left\{g^{il}\gamma_l^k+g^{ir}\gamma_r^k+\frac{2\eta}{1-2\eta}\,g^{ik}\gamma_l^l\right\}. \quad . \quad . \quad (11.7)$$

If the coordinate curves θ_i are along the principal directions of strain then γ_{ik}, g_{ik}, G_{ik}, g^{ik}, G^{ik} are all zero when $i \neq k$. Hence, when the stress–strain relations are of the form (11.6), $\tau^{ik}=0$ $(i \neq k)$, so that the principal directions of stress in the strained body B coincide with the principal directions of strain.

REFERENCES.

Biezeno, C. B., and Grammel, R., 1939, *Technische Dynamik*. Springer, Berlin.

Brillouin, L., 1938, *Les Tenseurs en Mécanique et en Élasticité,* Masson, Paris, p. 226.

Courant, R., and Hilbert, D., 1931, *Methoden der Mathematischen Physik.* Vol. 1. Springer, Berlin.

Deuker, E. A., 1941, *Deutsche Mathematik,* **5,** 546.

Gleyzal, A., 1949, *Quart. App. Math.,* **6,** 429.

Love, A. E. H., 1927, *A Treatise on the Mathematical Theory of Elasticity.* Cambridge University Press.

Milne-Thomson, L. M., 1948, *Proc. 7th International Congress for Applied Mechanics,* **33,**

Murnaghan, F. D., 1937, *Amer. Jour. Math.,* **59,** 233.

Neuber, H., 1943, *Z. angew. Math. Mech.,* **23,** 321.

Richter, H., 1948, *Z. angew. Math. Mech.,* **28,** 205.

Richter, H., 1949, *Z. angew. Math. Mech.,* **29,** 65.

Trefftz, E., 1928, *Handbuch der Physik,* Vol. 6. Springer, Berlin 1939.

---

† This result may also be obtained by assuming that the strain-energy function W is a function of the strain invariants. See, for example, F. D. Murnaghan (1937).

XXVII. *The Emission of Short-range Alpha Particles from Light Elements under Proton Bombardment.—II. Further Observations on the Reaction* ^{10}B($p\alpha$)^{7}Be.

By W. E. Burcham * and Joan M. Freeman †.

[Received December 23, 1949.]

Summary.

The energy release in the reaction ^{10}B($p\alpha$)^{7}Be has been measured, using targets of known thickness, and found to be $1{\cdot}147 \pm 0{\cdot}010$ keV. Evidence for the existence of an excited state of energy about 400 keV. in the ^{7}Be nucleus was obtained. The cross-section for the reaction was found by comparing the yield of alpha particles with the yield from the ^{10}B($n\alpha$)^{7}Li reaction observed with the same target. A value of $4{\cdot}5 \times 10^{-27}$ cm.^{2} at 330 keV. proton energy was found.

§ 1. Introduction.

In a previous paper (Burcham and Freeman 1949, subsequently referred to as I.) measurements of the energy release and cross-section for the reaction

$$^{10}\mathrm{B} + {}^1\mathrm{H} \rightarrow {}^7\mathrm{Be} + {}^4\mathrm{He} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

were reported. An improved cross-section determination has now been made by comparing the yield of reaction (1) with that of the slow neutron reaction

$$^{10}\mathrm{B} + {}^1\mathrm{n} \rightarrow {}^7\mathrm{Li} + {}^4\mathrm{He}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

This work, together with further observations on the energy release and excitation function for reaction (1), is described in the present paper.

§ 2. Energy Release in the Reaction ^{10}B($p\alpha$)^{7}Be.

(a) *Q-value for the ground-state transition.*

The energy release reported in I. was $1{\cdot}11 \pm 0{\cdot}06$ MeV., while in a similar investigation Chao, Lauritsen and Tollestrup (1949) found a value of $1{\cdot}148 \pm 0{\cdot}005$ MeV. The large error in the former determination arose partly from uncertainty in the thickness of the isotopic targets used, and in order to eliminate this error a special study of the alpha-particle group emitted from boron targets of different thickness has now been made. A set of targets each of area 0·20 cm.^{2}, and of thickness known from the ion current in the electromagnetic separator, was prepared ; the density of the deposits ranged from 30 to 300 micrograms per cm.^{2}. These

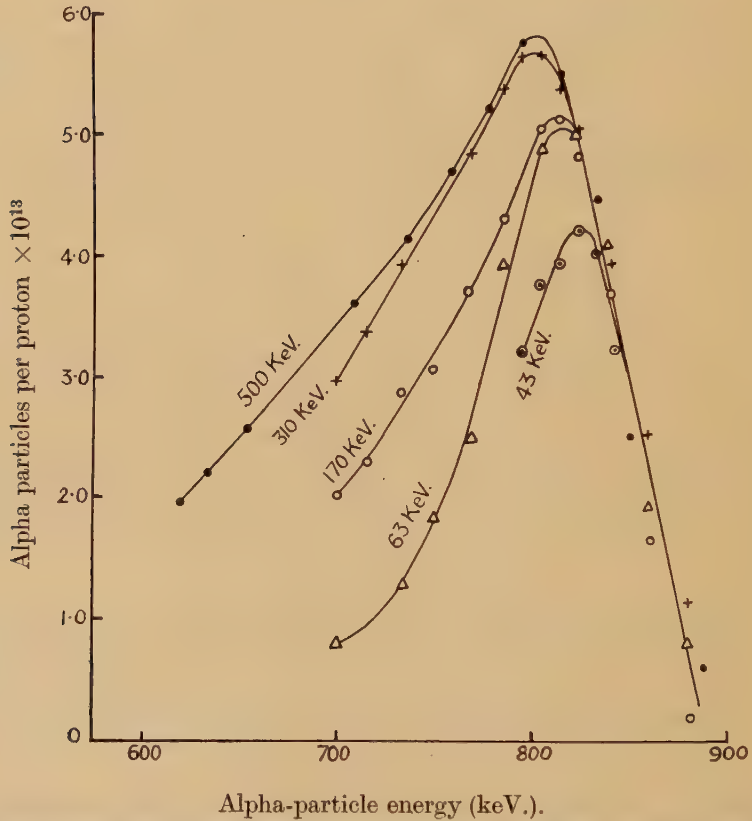
---

* Selwyn College, Cambridge ; † Newnham College, Cambridge ‡.

‡ Communicated by the Authors.

targets were bombarded with 432 keV. protons, and the alpha particles emitted at 123° to the proton beam were observed in the magnetic analyser as described in I. Some of the curves thus obtained are shown in fig. 1 ; the energy scale for these curves was obtained from a calibration of the magnetic analyser carried out by means of protons of energy known to ±5 keV., scattered from a thin gold film as described in I. In the experiments with boron a fluorescent screen of circular aperture

Fig. 1.

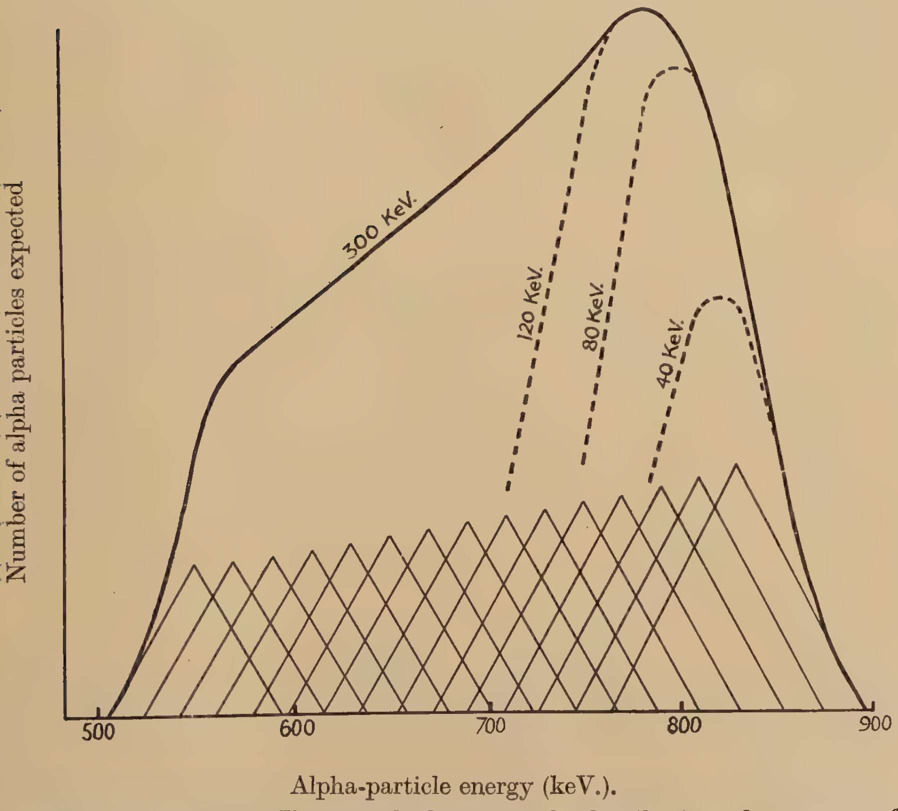


Alpha-particle energy (keV.).

Alpha-particles from the reaction $^{10}B(p\alpha)^{7}Be$. Proton energy 432 keV. ; angle of observation 123° ; target thicknesses 43, 63, 170, 310 and 500 keV. for alpha particles.

1·0 cm. in diameter was used, and the resolving power of the analyser, as defined in I., was 25 (width 66 keV. for alpha particles of energy 830 keV.). From the known analyser resolution and the excitation function for the reaction a curve representing the energy spectrum to be expected from a target of given thickness was constructed. For this purpose it was assumed that the alpha particles emerging from a particular depth in the target gave a spectrum which could be approximated by a triangle of height proportional to the cross-section for the reaction at the average proton energy for this depth in the target, and of width (at half-maximum) determined by the resolution of the analyser. A series of such triangles was drawn with 20 keV. intervals, the first representing alpha particles coming from an average depth of 10 keV., the second representing alpha particles from a depth of 30 keV., and so on, for a target of total thickness equivalent to 300 keV. (0·2 mgm./cm.^{2}), as shown in fig. 2. An integrated curve which would be expected to represent the

Fig. 2.



Reaction $^{10}B(p\alpha)^7Be$. Theoretical alpha-particle distributions for targets of different thickness.

observed spectrum was thus obtained (full curve). Similar curves, shown dotted in fig. 2, were drawn for targets of equivalent thickness 40, 80 and 120 keV. The intensity falls off on the left-hand side of the peak due to the effects of the exponential form of the excitation function and of the decreasing resolution width. It can be seen that for targets of equivalent thickness 120 keV. or more the peak of the spectrum occurs at an energy 60 keV. less than that for an infinitely thin target, and that only the first 120 keV. equivalent thickness of the target contributes to the peak.

The position and intensity of the latter should thus be independent of the thickness for such targets. For thinner targets the intensity would be expected to decrease and the peak to move to the right. A comparison of the constructed curves and the experimental curves of fig. 1 shows that the latter set, although somewhat narrower, exhibit the general features expected, and justifies the procedure of deducing from the constructed curves the depth in the target in keV. equivalent for which the peak of the alpha-particle group from each of the experimental targets would be expected to occur. The stopping power of the boron targets for alpha particles of about 800 keV. energy was assumed to be 1450 keV. per mgm./cm.^{2}, which is consistent with the data of Madsen and Venkateswarlu (1948) on the loss of energy of protons in beryllium*. The results are set out in Table I., which gives the actual target thickness in mgm./cm.^{2}, the effective thickness allowing for inclination θ of the target to the proton beam, the corresponding thickness in keV. equivalent for alpha particles, the depth in the target for which the peak would be expected to occur, and the alpha-particle intensity and energy observed at the peak of each group.

Table I.

Target thickness			Depth of peak from surface (keV.) (theoretical)	Intensity at peak (observed) alpha-particles per 10^{13} protons	Energy at peak (observed) (keV.)
mgm./cm.^{2}	$\frac{\text{mgm./cm.}^2}{\cos\theta}$	keV. equiv.			
0·300	0·345	500	60	5·7	805
0·185	0·215	310	60	5·6	804
0·165	0·190	270	60	4·9	814
0·105	0·120	170	60	5·1	815
0·038	0·044	63	32	5·0	819
0·026	0·030	43	22	4·2	825

When the alpha-particle energy is plotted against the theoretical depth of the peak from the surface (column 4) the result shown in fig. 3 is obtained. The extrapolation to zero target thickness gives an alpha-particle energy of 833 keV., with an uncertainty in the extrapolation of 3 keV. This figure must be further corrected to allow for the passage of the alpha particles through the thin layer of carbon deposited on the surface of the targets from pump-oil vapours. The thickness of this layer was found by determining the yield of radioactive N^{13}, formed in the C^{12}($p\gamma$) reaction, from the boron targets in comparison with that obtained from a thick carbon target. The average value of the thickness found for the boron targets was equivalent to 3±1 keV. for protons ; this correction has been made to the incident proton energy and a correction of 12±4 keV. to the emergent alpha-particle energy. With the resulting proton energy of 429±7 keV. and alpha-particle energy of 845±7 keV. at 123° the Q-value for the ground state transition in the $^{10}B(p\alpha)$ reaction is found to be

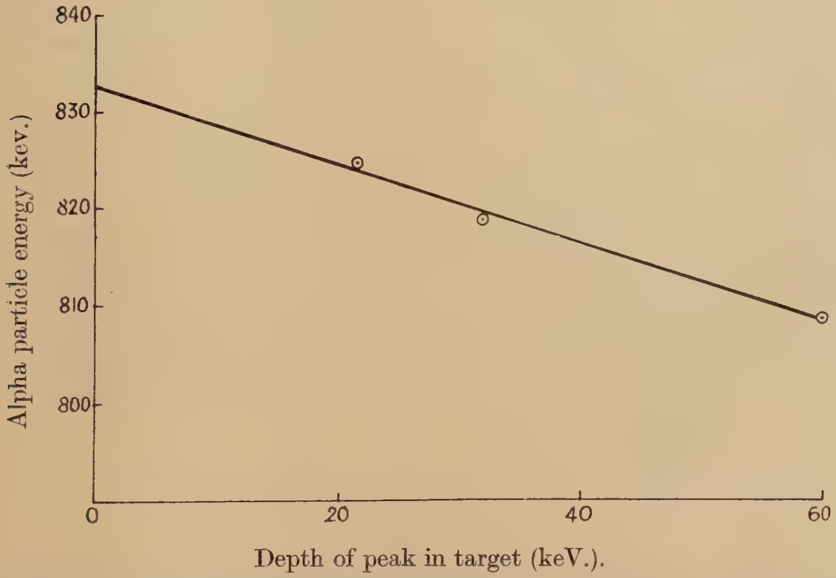
---

* *Note added in proof.*—The stopping power of beryllium for protons of velocity equal to that of 800 keV. alpha particles can be obtained directly from the recent work of Warsaw (1949, *Phys. Rev.*, **76**, 1759).

$$Q=1{\cdot}147\pm0{\cdot}010 \text{ keV.},$$

in good agreement with the work of Chao, Lauritsen and Tollestrup (1949). The error in this Q-value is less than that calculated directly from the stated errors in the alpha-particle and proton energies, since calibration errors affect these two energies in the same sense.

Fig. 3.



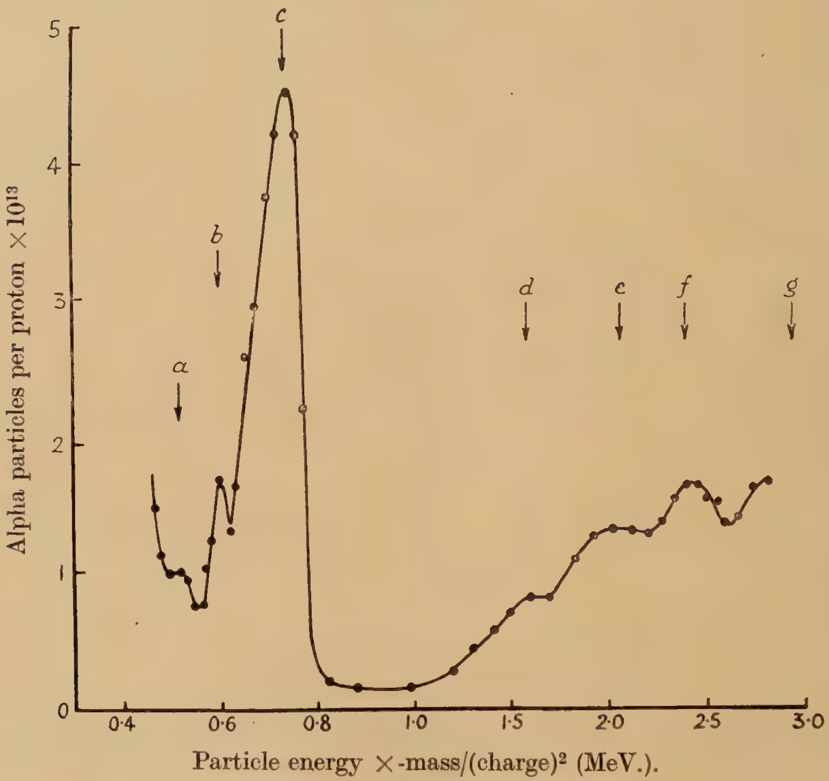
Reaction $^{10}B(p\alpha)^7Be$. Extrapolation of observed alpha particle energy to zero target thickness.

The curves of fig. 1 show that the shape of the high energy side of the alpha-particle distribution is independent of target thickness. An estimate of the Q-value from an "extrapolated" energy derived by drawing a tangent to the observed spectrum gave 1·157±0·020 MeV. ; this is less accurate than the preceding value owing to greater uncertainty in the analyser calibration for "extrapolated" ranges.

(b) *The first excited state of* ^7Be.

It was pointed out in I. that the similarity between the nuclei ^7Li and ^7Be suggested that there might be a low-lying state of ^7Be analogous to the 480 keV. state of ^7Li. A search (I. fig. 10) for alpha particles emitted in the $^{10}B(p\alpha)^7Be$ reaction which would correspond to transitions to such an excited state showed that the intensity of such a group was less than 10 per cent of that of the main group for a proton energy of 230 keV. If, however, the *a priori* probability of emission of the two groups is the same, barrier penetrability would reduce the intensity of an alpha-particle group corresponding to an excited state of more than 400 MeV. excitation to about this fraction of the intensity of main group. The use of a much higher bombarding voltage to increase the yield was impossible since the alpha particles would have been obscured by protons scattered from the target backing of thick copper, and targets of the ^{10}B isotope deposited on thin foils were not available. In order to extend the work to higher voltages a search was made for singly charged alpha particles corresponding to the excited state transition ; fig. 4 shows the results obtained with a target of thickness 0·165 mgm./cm.^{2}, a proton energy of 430 keV., an angle of observation of 135°, and an analyser resolving power of 50. The analyser was calibrated at the higher energy values by observations of the protons from the ^{12}C$(dp)^{13}$C reaction (Q=2·70 MeV.). The distribution

Fig. 4.



Reaction ^{10}B$(p\alpha)^7$Be. Singly and doubly charged groups observed at 135° with E$_p$=430 keV. and resolution of 4 per cent in energy. Note change of scale at 1·0 MeV.

of particles observed with high analyser fields could not be completely resolved into separate groups, owing probably to the retardation of the recoil groups in the target. If, however, correction is made, as described in § 2, for the effective depth of the target and for the presence of a carbon layer, the expected positions of the singly and doubly charged alpha particles and singly and doubly charged ^{7}Be nuclei corresponding to the ground state transition are as marked (g, c, f, b). The groups (e, d) could be due to singly charged alpha particles and Be nuclei corresponding to a transition to an excited state of about 400 keV., and there is some slight evidence for an associated ^{4}He^{++} group (a). In confirmation of this interpretation it was shown that all the groups except c and g (of energy 740 keV.) disappeared when the zinc sulphide screen was covered by a 1·5 mm. foil, which indicated that groups a, b, d, e, f were of lower energy. Groups f and e were found to move with increasing bombarding proton energy in the way expected of ^{7}Be$^+$ and ^{4}He$^+$ nuclei respectively.

At this time it was learned that definite evidence for the existence of a 435 keV. state in ^{7}Be had been obtained by Brown, Chao, Fowler and Lauritsen* with better experimental conditions, and it was felt that nothing further could be gained by continuing work on the low energy group with the available targets. The evidence for the existence of an excited state of ^{7}Be near 400 keV. is in agreement with the results of Brown *et al.*; the ratio of intensities of the transitions to the ground and excited states for a proton energy of 430 keV. cannot be estimated accurately from the present results, but appears to be about 5 to 1.

The existence of a 435 keV. state in ^{7}Be should lead to the emission of two groups of neutrons in the reactions

$$^7\text{Li}(pn)^7\text{Be} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

and

$$^6\text{Li}(dn)^7\text{Be} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

and to gamma radiation of energy 430 keV. in these two reactions and in the ^{10}B($p\alpha$)^{7}Be reaction. Gamma radiation of this energy has been observed in the ($p\alpha$) reaction and in reaction (4) by Lauritsen and Thomas*, and in the ($p\alpha$) reaction in this laboratory by Jelley and Freeman (unpublished). The observations of Freier, Lampi and Williams (1949) and of Taschek and Hemmendinger (1948) revealed no second neutron group in reaction (3) in intensity exceeding 10 per cent of the main group. Recently, however, evidence for neutron groups in this reaction corresponding to levels in ^{7}Be of energies 0·206±0·07, 0·470±0·07 and 0·745±0·07 has been obtained by Grosskreutz and Mather (1949). The 745 keV. state would not have been observed in the present ($p\alpha$) experiments, and the 470 keV. state seems to correspond to that found by Brown *et al.* and also indicated in the present work. Alpha particles corresponding to a transition to a 205 keV. state would have been observed in approximately the position occupied by group *b*, which was interpreted as $^7Be^{++}$ recoils in fig. 4. The present work does not definitely establish the identity of group *b*, but group *f*, which was interpreted as the corresponding $^7Be^+$ group, was shown to increase in energy with increasing bombarding energy as expected for particles of this mass and charge.

---

* We are indebted to Professor T. Lauritsen for sending us this information in advance of publication.

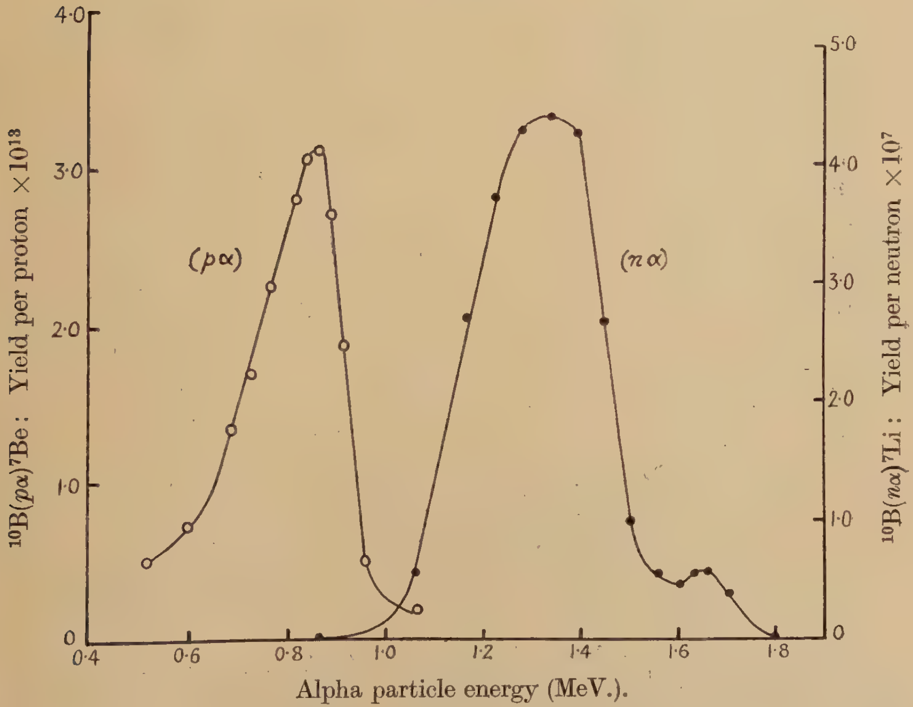
Evidence for a second neutron group in reaction (4) has been obtained in the Cavendish Laboratory by Gibson (in course of publication) using the photographic plate technique. The neutron spectrum is consistent with an energy level of 435 keV. No evidence for a 745 keV. level was obtained by Mandeville, Swan and Snowdon (1949) in their study of this reaction.

§ 3. COMPARISON OF THE REACTIONS $^{10}B(p\alpha)^7Be$, $^{10}B(n\alpha)^7Li$.

(a) *Energy release.*

For the observation of reaction (2) the magnetic analyser and target chamber were transported to the Atomic Energy Research Establishment, Harwell, and set up in such a way that a beam of thermal neutrons could pass through the ^{10}B target. The region of the target of area 0·31 cm.^{2} which had previously been bombarded by protons was defined by a slight discoloration and a screen was placed over the target for the $(n\alpha)$ experiments, so that only the alpha particles emitted from the area thus defined could be observed. A foil of stopping power 4 mm. was placed over the zinc sulphide screen to prevent the detection of the $^7Li^{++}$ nuclei produced in the reaction, while not appreciably reducing the efficiency for alpha-particle detection. The analyser resolving power was 25 and an angle of observation of 83° was used. The energy spectrum obtained is shown in fig. 5, curve $(n\alpha)$. The two alpha-particle groups, corresponding to transitions to the ground state and to the first excited state, are clearly evident. The main group in the $(n\alpha)$ experiment is appreciably broader than that found in $(p\alpha)$ experiments, and this can be explained on the basis of target thickness effect by reference to a curve constructed by a procedure similar to that described for the derivation of fig. 2. In the $(n\alpha)$ case the cross-section for the reaction is the same at all depths in the target, so that the triangles representing the contribution to the total spectrum of the alpha-particles arising from various depths all have the same height, decreasing in width only with decreasing energy. The curve constructed on this basis is shown in fig. 6. It is seen that contributions to the peak are received from alpha particles emitted in the first 200 keV. equivalent thickness of the target, and the peak occurs at an energy 110 keV. below that of alpha particles emitted from the surface. A comparison of this curve with that of fig. 5 shows that, in the position used, the target was effectively about 200 keV. " thick ", and that a correction of 110 keV. should therefore be applied to the energy at the peak of the spectrum. The energy of the main alpha-particle group emitted from the surface of the target was thus deduced as 1·45±0·02 MeV. The effective thickness of the carbon film on the target was estimated, as previously described, to be 20 keV. for alpha particles, giving a true particle energy of 1·47±0·02 MeV. This gave the energy release in the $^{10}B(n\alpha)^{7}Li^{*}$ reaction as Q=2·31±0·04 MeV. From the position of the alpha-particle group due to the ground state transition the true energy of these particles was estimated to be 1·78±0·03 MeV., giving an energy release of 2·80±0·05 MeV. These values are in agreement with previous work (Boggild 1945, Gilbert 1948, Chao, Lauritsen and Tollestrup 1949). The accuracy of the present determination was limited by the intensity of the slow neutron beams available and by the solid angle of the analyser.

Fig. 5.



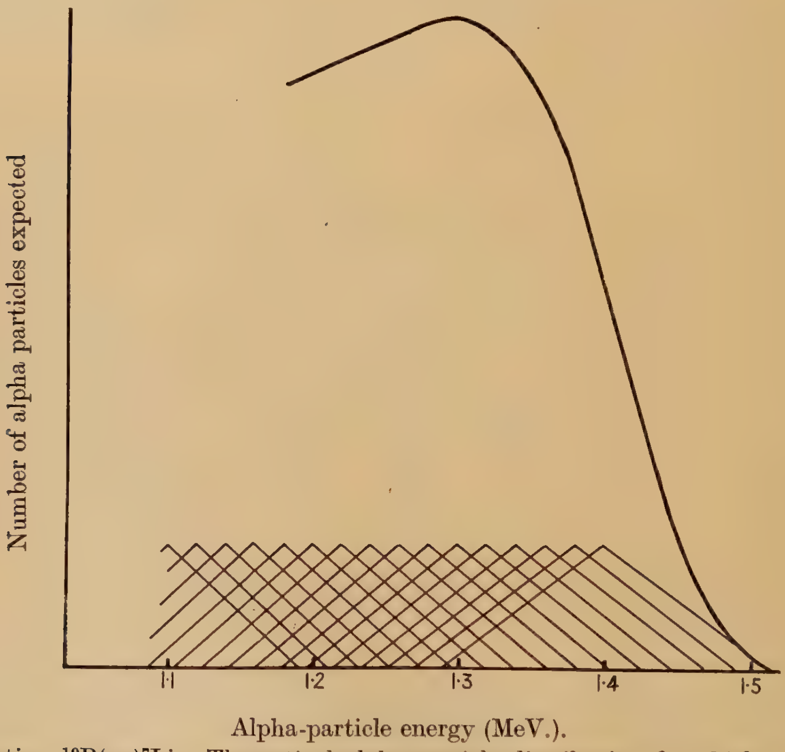
Reactions $^{10}B(p\alpha)^{7}Be$ and $^{10}B(n\alpha)^{7}Li$. Alpha-particle groups.

The ^{10}B target used in this (nα) experiment was bombarded with protons with the main object of comparing the yields from the two reactions, but the results also allowed another check on the energy release. A proton energy of 330 keV. was used, and an angle of observation of 83°. The energy spectrum (fig. 5, curve (pα)) indicated that the effective target thickness was greater than 120 keV., as expected from the (nα) result, so that a correction of 60 keV. for target thickness was necessary. An allowance of 20 keV. was made for the carbon film formed on the target, and the true alpha-particle energy was deduced as 935±16 keV. This gave a value of 1·14±0·03 MeV. for the energy release, in satisfactory agreement with the more accurate estimate given above.

(b) *Cross-sections.*

The ordinates of fig. 5 give directly the yield of alpha particles in the reactions (1) and (2) per incident proton or thermal neutron respectively. The yield for the proton reaction was obtained by integrating the charge collected on the target during the ($p\alpha$) experiments, care being taken to eliminate the effects of secondary electrons. The amplifier gain and bias setting were known (see I., § 2 (c)) to be suitable for the detection of alpha particles of 800 keV. with 100 per cent efficiency, and were high enough to make correction for the presence of the lower energy $^7Be^{++}$ nuclei unnecessary. In the ($n\alpha$) experiment the flux of neutrons incident on the target was measured by placing a calibrated manganese foil of area 1 cm.^{2} in the position of the target and allowing it to be irradiated for a known period. The beta-activity was found in a standard counting arrangement, whose efficiency was checked by means of a uranium source ; the neutron flux used was $6·5\times10^6\pm5$ per cent. No correction

Fig. 6.

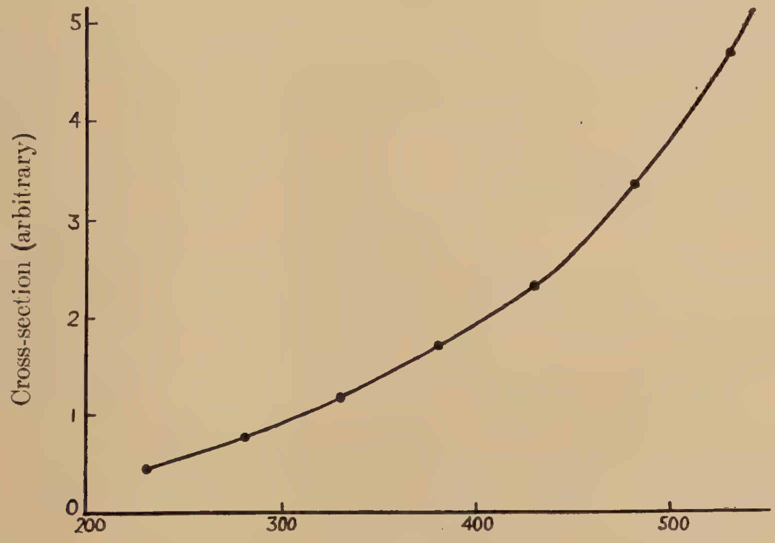


Reaction $^{10}B(n\alpha)^7Li$. Theoretical alpha-particle distribution for thick target.

for the presence of ^{7}Li recoils in reaction (2) was necessary, since an absorbing foil was used over the zinc sulphide screen in the $(n\alpha)$ experiment.

The analyser resolution in both $(n\alpha)$ and $(p\alpha)$ experiments was less than the equivalent thickness of the target. The ratio of yields must therefore be corrected by a factor (1·57) equal to the ratio of the energies of the particle groups. A further correction for charge exchange of 1·27 for the $(n\alpha)$ and 1·66 for the $(p\alpha)$ reaction was deduced from the figures given by Briggs (1927). With these corrections, and after subtraction of background counts, the ratio of yields at 83° per incident particle becomes $1{\cdot}37\times10^{-6}$. This, assuming isotropic angular distributions, is taken to be the ratio of the cross-section for reaction (1) with 330 keV. protons to that of reaction (2) with thermal neutrons. The neutron flux measurement by means of the $1/v$-law manganese detector assumed a neutron velocity of $2{\cdot}5\times10^5$ cm. per sec., and the ^{10}B cross-section for neutrons of this velocity is 3285 barns, from the results of Fermi, Marshall and Marshall (1947). The $(p\alpha)$ cross-section is, therefore,

Fig. 7.



Proton energy (keV.).

Reaction ^{10}B$(n\alpha)^7$Be excitation function. Absolute cross-section at 330 keV.$=4{\cdot}5\times10^{-27}$ cm.^{2}

$$\sigma_{p\alpha}\,(330\text{ keV.})=4{\cdot}5\times10^{-27}\text{ cm.}^2$$

In order to compare this figure with that previously given for a proton energy of 530 keV. in I., an extension of the excitation function given in I., fig. 11, was made. This is given in fig. 7, the ordinate scale being fixed by the 330 keV. determination, and from this

$$\sigma_{p\alpha}\,(530\text{ keV.})=1{\cdot}9\times10^{-26}\text{ cm.}^2,$$

which is to be compared with the value of $1{\cdot}2 \times 10^{-26}$ cm.^{2} given in I. The difference is not surprising in view of the uncertainty of the factors which entered into the earlier value ; in the present method knowledge of the solid angle of the analyser is not required, and the exact target thickness need not be known since the analyser selects a definite layer of the target determined by the resolution. The calculation of the cross section given in I. from the yields shown as ordinates in I., fig. 11, included a large correction for the ratio of areas of the proton beam and the boron target; this correction is unnecessary in the present experiment. The main source of experimental error, which has not been eliminated in the present cross-section determination, is a possible over-estimate of the counting efficiency for alpha particles of 3·5 mm. range emerging from the stopping foil in the $(n\alpha)$ reaction. From the curves given in I., fig. 4 (*b*) (bias 20), this error is unlikely to exceed 20 per cent.

As a further check the cross-section for the $(p\alpha)$ reaction was also deduced from the observations on small weighed targets described in § 2. The solid angle of the analyser and the target depth determined by resolution were assumed and a value of $2{\cdot}0 \times 10^{-26}$ cm.^{2} at $E_p = 530$ keV. was found, in agreement with that deduced from the $(n\alpha)$ reaction within the limits of error of the measurements.

ACKNOWLEDGMENTS.

We wish to thank Sir John Cockcroft for granting us permission to use the facilities at the Atomic Energy Research Establishment, Harwell, and Mr. F. W. Fenning for much assistance there. We are grateful to Dr. W. D. Allen, of the Atomic Energy Research Establishment, Harwell, for the boron isotopes. We also wish to thank Messrs. W. Birtwhistle and D. D. Stewart for much technical assistance.

REFERENCES.

BETHE, H. A., 1937, *Rev. Mod. Phys.*, **9,** 69.
BOGGILD, J. K., 1945, *Kgl. Dansk. Vidensk. Selsk. Mat.-fys. Medd.*, **23,** No. 4.
BRIGGS, G. H., 1927, *Proc. Roy. Soc.* A, **114,** 341.
BURCHAM, W. E., and FREEMAN, J. M., 1949, *Phil. Mag.*, **40,** 807.
CHAO, C. Y., LAURITSEN, C. G., and TOLLESTRUP, A. V., 1949, *Phys. Rev.*, **76,** 586.
FERMI, E., MARSHALL, J., and MARSHALL, L., 1947, *Phys. Rev.*, **72,** 193, 703 *b*.
FREIER, G., LAMPI, E. E., and WILLIAMS, J. H., 1949, *Phys. Rev.*, **75,** 901.
GILBERT, C. W., 1948, *Proc. Camb. Phil. Soc.*, **44,** 447.
GROSSKREUTZ, J. C., and MATHER, K. B., 1949, *Bull, Am. Phys. Soc.*, **24,** No. 7, 15.
MADSEN, C. B., and VENKATESWARLU, P., 1948, *Phys. Rev.*, **74,** 648.
MANDEVILLE, C. E., SWAN, C. P., and SNOWDON, S. C., 1949, *Phys. Rev.*, **76,** 980.
TASCHEK, R., and HEMMENDINGER, A., 1948, *Phys. Rev.*, **74,** 373.

XXVIII. *Relaxation Methods applied to Problems of Plastic Flow* *.— I. *Notched Bar under Tension.*

By J. A. Jacobs,
Royal Holloway College, University of London †.

[Received January 3, 1950.]

Summary.

Orthodox analysis of most problems of plastic flow is extremely difficult, in that the governing equations are not linear, so that partial solutions may not be superposed. Many problems, however, yield to treatment by relaxation methods. This paper considers the consequence of a particular hypothesis regarding plastic strain, viz. the Mises-Hencky criterion, and, by application of a numerical method due to Allen and Southwell, traces the effect of plastic straining. The success of the method in no way depends upon the particular yield hypothesis adopted. Three problems are considered:

(i) A two-dimensional slit tension member.
(ii) A sharp-ended bar under specified edge tractions.
(iii) The uniform compression of a block between two smooth, rigid plates which do not overlap it.

In each case, results and conclusions are summarized and presented graphically.

§ 1. Introduction.

Orthodox analysis of most problems of plastic flow is extremely difficult in that the governing equations are not linear, so that partial solutions may not be superposed. Moreover, in many cases the shape of the boundary and the boundary conditions are such as to preclude an exact solution, even in the elastic case. Many of these difficulties are resolved when the problem is attacked by relaxation methods—thus, for example, the shape of the boundary does not materially increase the complexity of the problem. Again, many of the exact solutions are extremely complicated, being usually expressed in series form. The convergence is often very slow, and the numerical value of the stresses at various points would require as much labour as a normal solution by relaxation methods.

Relaxation methods are peculiarly suited to a class of problem involving boundaries or interfaces not initially known—problems which, in many cases, present insuperable difficulties to an orthodox approach. In this connection, besides problems of plastic flow, relaxation methods have been successfully applied by Shaw and Southwell (1941) to problems relating to the percolation of fluids through porous materials, and by Southwell and Vaisey (1946) to fluid motions characterized by free streamlines.

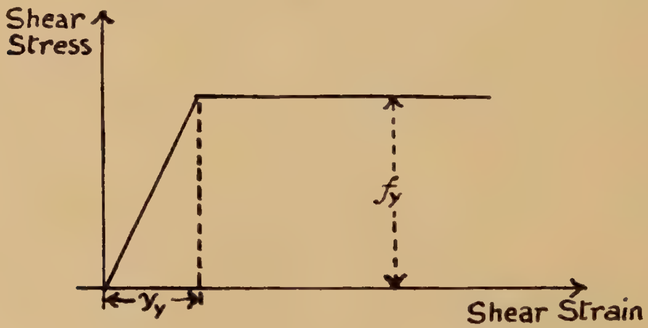
---

* This paper is based upon work forming part of the writer's Ph.D. thesis, London University, June 1949.

† Communicated by the Author.

The true criterion of elastic failure and the relation between stress and strain during plastic distortion in real materials has yet to be determined. Many hypotheses have been advanced, the validity of any of which only experiment can finally endorse. Direct experimental study of stress-strain relations is made more difficult by the fact that stress is not itself measurable. Experiment can only verify or deny results computed from one particular theory in regard to total displacements produced by imposed external tractions. We adopt a particular hypothesis regarding plastic failure and use it as a basis for our computations. It is to be emphasized that the methods evolved do not depend on the particular hypothesis adopted, but could be applied similarly with any other criterion.

Fig. 1.



§ 2. GENERAL THEORY.

The first attack on plastic problems by relaxation methods was made by Christopherson and Southwell (1938), who considered the plastic torsion of a bar of equilateral triangular cross-section. Later Christopherson (1940) applied the same methods to I-section girders, and Shaw (1942) considered hollow (*i.e.* multiply-connected) sections, where plastic strain can commence at the external or at an internal boundary. All these problems of plastic torsion based their solution on a hypothesis put forward by Prandtl (1923) which assumes the relation between shear stress and shear strain to be that indicated in fig. 1.

For stresses below some limiting value f_y the material behaves elastically, but the stress cannot exceed this limiting value f_y with which any shear strain in excess of a limiting elastic shear strain γ_y can be associated. This imposes (in the torsion problem) an upper limit upon the gradient of the stress function. This restriction can be represented by what is usually known as the Prandtl roof, and the ensuing computation is straightforward.

The next major advance was made by Allen and Southwell (1950), who investigated more complex problems of plastic straining in two-dimensional stress systems. In particular, they considered notched tensile specimens having a central waist. Two examples were taken, in which the notches were semicircular and sharp re-entrant right angles. One of the examples we have considered is a limiting case of this type of problem—a slit tension member.

For the more complex stress systems considered, a generalization of the Prandtl hypothesis is required. The generalization we adopt rests on two assumptions :

(i) The first occurrence of plastic straining is determined by the " Mises-Hencky " criterion, *i.e.* the normal elastic relations hold until the principal stresses p_1, p_2, p_3 attain such values that

$$(p_2-p_3)^2+(p_3-p_1)^2+(p_1-p_2)^2=6k^2, \quad . \quad . \quad . \quad . \quad (1)$$

k being a constant for any material. The relation (1) holds thereafter throughout any region of plastic strain.

(ii) Where the material has yielded plastically, the strain consists of both an elastic and a plastic component. For the elastic part the usual stress-strain relations hold, viz.,

$$\mathbf{e}_1=\frac{1}{\mathrm{E}}\{p_1-\sigma(p_2+p_3)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

etc., where E (Young's modulus) and σ (Poisson's ratio) have normal values.

The plastic part is assumed to be equivoluminal, *i.e.*

$$\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and in it the incremental shear strains resulting from the three principal shear stresses are directly proportional to these stresses and occur on the same three planes, *i.e.* if q_1, q_2, q_3 denote the prinicpal shear stresses, and $\Delta\gamma_1$, $\Delta\gamma_2$, $\Delta\gamma_3$ the principal incremental shear strains, then

$$\Delta\gamma_1:\Delta\gamma_2:\Delta\gamma_3=q_1:q_2:q_3. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Then if $\Delta\mathbf{e}_1$, $\Delta\mathbf{e}_2$, $\Delta\mathbf{e}_3$ are the principal incremental plastic strains, (4) may be written

$$(\Delta\mathbf{e}_2-\Delta\mathbf{e}_3):(\Delta\mathbf{e}_3-\Delta\mathbf{e}_1):(\Delta\mathbf{e}_1-\Delta\mathbf{e}_2)=(p_2-p_3):(p_3-p_1):(p_1-p_2). \quad (5)$$

where $\Delta\mathbf{e}_1$, $\Delta\mathbf{e}_2$, $\Delta\mathbf{e}_3$ have the same directions as p_1, p_2, p_3. Equations (4) and (5) may be written

$$\frac{\Delta\gamma_1}{q_1}=\frac{\Delta\gamma_2}{q_2}=\frac{\Delta\gamma_3}{q_3}=3\Delta\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$\frac{(\Delta\mathbf{e}_2-\Delta\mathbf{e}_3)}{p_2-p_3}=\frac{(\Delta\mathbf{e}_3-\Delta\mathbf{e}_1)}{p_3-p_1}=\frac{(\Delta\mathbf{e}_1-\Delta\mathbf{e}_2)}{p_1-p_2}=\tfrac{3}{2}\Delta\lambda, \quad . \quad . \quad (7)$$

where $\Delta\lambda$ is an incremental quantity.

Thus $$2\Delta\mathbf{e}_1-\Delta\mathbf{e}_2-\Delta\mathbf{e}_3=\tfrac{3}{2}\Delta\lambda(2p_1-p_2-p_3),$$

i. e., using (3),

$$\Delta\mathbf{e}_1=\Delta\lambda\{p_1-\tfrac{1}{2}(p_2+p_3)\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus, as regards the plastic component of the distortion, the stress-*incremental* strain relations have normal forms, the effective values of Young's modulus and Poisson's ratio being $\frac{1}{\Delta\lambda}$ and $\frac{1}{2}$ respectively.

The total strains associated with given stresses are given by

$$e_1=\mathbf{e}_1+\mathbf{e}_1$$

etc. That an exact analysis is rendered difficult is shown from the fact that (2) relates to *total* and (8) to *incremental* strains, both expressible in terms of total stresses.

Combining (2) and (8) gives three equations of the type

$$\Delta e_1=\frac{1}{\mathrm{E}}\{\Delta p_1-\sigma(\Delta p_2+p_3)\}+\Delta\lambda\{p_1-\tfrac{1}{2}(p_2+p_3)\}. \quad . \quad . \quad . \quad (9)$$

Allen and Southwell (1950) deduce from the Mises-Hencky criterion (1) that for plane stress

$$\mathrm{X}_x^2-\mathrm{X}_x\mathrm{Y}_y+\mathrm{Y}_y^2+3\mathrm{X}_y^2\leqslant 3k^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

whilst for plane strain

$$(\mathrm{X}_x-\mathrm{Y}_y)^2+4\mathrm{X}_y^2\leqslant 4k^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the sign of inequality relating to elastic conditions and the sign of equality to conditions of plastic straining. The condition (10) follows immediately, but (11) is obtained on the simplifying assumption that $\sigma=\frac{1}{2}$, whatever value may be appropriate in (9). The error due to this assumption is small.

§ 3. THE COMPUTATION OF THE STRESSES.

In plane strain (and in plane stress), provided that there are no body forces, all of the conditions of stress-equilibrium may be satisfied by writing

$$\mathrm{X}_x=\frac{\partial^2\chi}{\partial y^2}, \quad \mathrm{X}_y=\frac{-\partial^2\chi}{\partial x\,\partial y}, \quad \mathrm{Y}_y=\frac{\partial^2\chi}{\partial x^2} \quad . \quad . \quad . \quad . \quad (12)$$

whatever be the nature of the accompanying strains. When known tractions act at a single boundary, χ and $\frac{\partial\chi}{\partial\nu}$ have the same boundary values whether plastic straining has occurred or not.

For specified edge tractions N_ν, N_s we have that

$$N_\nu \cos(x,\nu) - N_s \sin(x,\nu) = X_\nu$$

$$= X_x \cos(x,\nu) + X_y \sin(x,\nu)$$

$$= \left\{ \cos(x,\nu)\frac{\partial}{\partial y} - \sin(x,\nu)\frac{\partial}{\partial x} \right\} \frac{\partial \chi}{\partial y} = \frac{\partial}{\partial s}\left(\frac{\partial \chi}{\partial y}\right).$$

In a similar manner

$$N_\nu \sin(x,\nu) + N_s \cos(x,\nu) = Y_\nu$$

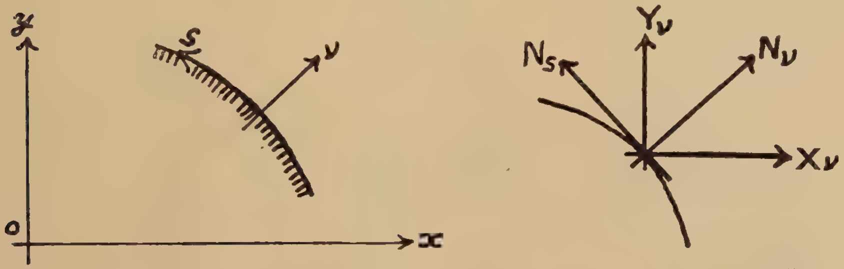
$$= Y_y \sin(x,\nu) + X_y \cos(x,\nu) = -\frac{\partial}{\partial s}\left(\frac{\partial \chi}{\partial x}\right).$$

Hence for all points of the boundary

$$-\frac{\partial \chi}{\partial x} = \int Y_\nu \, ds, \qquad \frac{\partial \chi}{\partial y} = \int X_\nu \, ds, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

both quantities being single-valued.

Fig. 2.



The specified boundary tractions can thus be reduced to given boundary values of $\frac{\partial \chi}{\partial x}$, $\frac{\partial \chi}{\partial y}$. Since the stress components depend on the derivatives of $\frac{\partial \chi}{\partial x}$, $\frac{\partial \chi}{\partial y}$, the lower limits of integration in (13) are arbitrary (so that $\frac{\partial \chi}{\partial x}$, $\frac{\partial \chi}{\partial y}$ may be given the value zero at some convenient point of the boundary).

Finally χ can be calculated at all points of the boundary from the equation

$$\chi = \int \left\{ \frac{\partial \chi}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial \chi}{\partial y} \cdot \frac{dy}{ds} \right\} ds, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

the lower limit again being arbitrary.

Writing (14) in the form

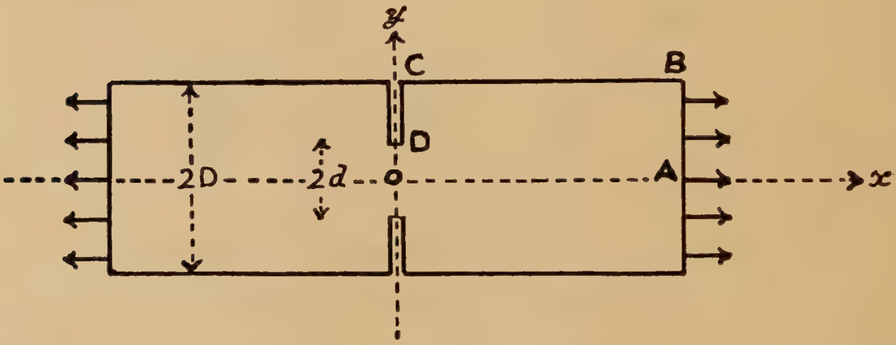
$$\chi = x\frac{\partial \chi}{\partial x} + y\frac{\partial \chi}{\partial y} - \int \left\{ x\frac{\partial^2 \chi}{\partial s\,\partial x} + y\frac{\partial^2 \chi}{\partial s\,\partial y} \right\} ds$$

$$= x\frac{\partial \chi}{\partial x} + y\frac{\partial \chi}{\partial y} + \int (xY_\nu - yX_\nu)\, ds, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

it follows that χ is single-valued at all points of the boundary.

Equations (13) and (15) enable boundary values of $\chi, \frac{\partial\chi}{\partial x}, \frac{\partial\chi}{\partial y}$ to be computed from specified edge tractions. So long as the whole of the plate remains elastic, the determination of χ at internal points is a biharmonic problem, whose method of solution by relaxation methods is given by Fox and Southwell (1945). Apart from a multiplying factor which increases linearly with the total load, the solution is unique. By substitution in the criterion (11) a value can be found for the greatest load that can be sustained elastically. If the load is increased still further, the criterion will be violated in some regions. In such regions the elastic solution must be modified to satisfy it, whilst leaving the values of $\chi, \frac{\partial\chi}{\partial x}, \frac{\partial\chi}{\partial y}$ unchanged on the boundary. This modification can be carried out by a relaxation method, described by Allen and Southwell (1950).

Fig. 3.



§4. PROBLEM (i.) STRESS DISTRIBUTION IN A TWO-DIMENSIONAL SLIT TENSION SPECIMEN.

Fig. 3 shows the slit tension specimen, subject to specified edge tensions. Three cases are considered, in which the ratio $\frac{D}{d}$, of the total width of the bar to the width at the waist is given the values 8, 4 and 2. From symmetry it will be sufficient to consider the quadrant OABCD. We can eliminate dimensional factors by writing

$$x = Dx', \qquad y = Dy', \qquad \chi = TD\chi', \quad . \quad . \quad . \quad . \quad . \quad (16)$$

2T denoting the total tension on the specimen.

Then such quantities as $\frac{\partial^2\chi'}{\partial x'^2}$ measure stresses as multiples of $\frac{T}{D}$, the average tensile stress at the ends. Hereafter, however, the dashes attached to x, y and χ will be suppressed, and the symbols will stand for purely numerical quantities.

$$\left.\begin{array}{l} \text{The boundary conditions are, along AB,} \\ \qquad X_x=\frac{\partial^2\chi}{\partial y^2} \\ \text{is specified (and was taken as 3200 to obviate decimals).} \\ \qquad \mathrm{X}_y=\frac{-\partial^2\chi}{\partial x\,\partial y}=0 \end{array}\right\} \quad . \quad (17)$$

along BC and CD, $\mathrm{X}_\nu=\mathrm{Y}_\nu=0$, *i.e.*

$$\frac{\partial}{\partial s}\left(\frac{\partial\chi}{\partial x}\right)=\frac{\partial}{\partial s}\left(\frac{\partial x}{\partial y}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

so that $\frac{\partial\chi}{\partial x}$ and $\frac{\partial\chi}{\partial y}$ are both constant.

If we take $\frac{\partial\chi}{\partial x}=0$ at D, then $\frac{\partial\chi}{\partial x}=0$ along DC and CB, and hence $\frac{\partial\chi}{\partial x}=0$ along AB (since $\frac{\partial\chi}{\partial x}$ is constant along BA and is thus equal to its value at B).

Since $\frac{\partial^2\chi}{\partial y^2}=3200$ along AB, $\frac{\partial\chi}{\partial y}=3200y$, taking $\frac{\partial\chi}{\partial y}=0$ at A. Thus at B, $(y=1)$, $\frac{\partial\chi}{\partial y}=3200$, and hence $\frac{\partial\chi}{\partial y}=3200$ along BC and CD.

Further, taking $\chi=0$ at A,

$$\chi=1600y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

along AB. Finally, from (14),

$$\chi=\int\left\{\frac{\partial\chi}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial\chi}{\partial y}\cdot\frac{dy}{ds}\right\}ds$$

$$=\int\frac{\partial\chi}{\partial y}\cdot\frac{dy}{ds}ds$$

along BCD where $\frac{\partial\chi}{\partial x}=0$

$$=\int 3200\,dy$$

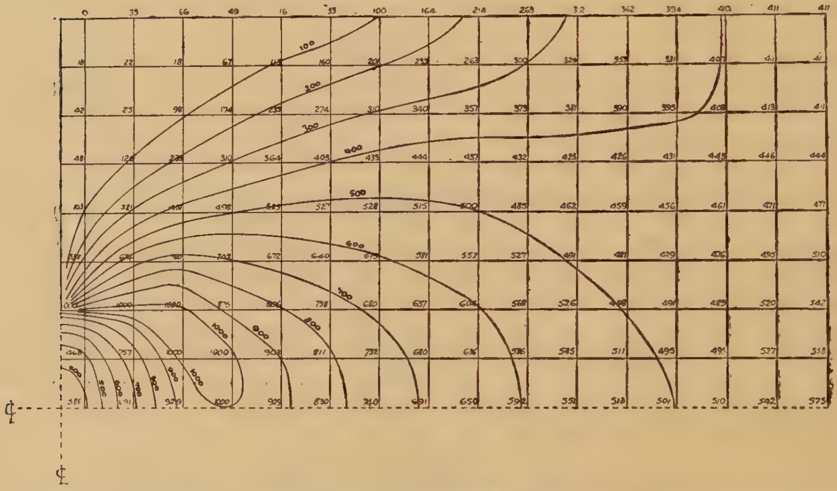
$$=3200y-1600 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

since $\chi=1600$ at B $(y=1)$. Actually along BC $(y=1)$, $\chi=1600$, and along CD, $\chi=3200y-1600$.

The case $\frac{\mathrm{D}}{d}=4$ was first attacked. After a preliminary investigation on a coarse mesh, a finer net $(h=\frac{1}{8})$ was used.

The computation of χ and the stresses in the elastic case was carried out by the relaxational methods given by Fox and Southwell (1945). From this elastic solution we have to determine what modifications are necessary when, in a region of plastic straining, the condition (11) supervenes. It appears that a plastic enclave * will start at the bottom of the slit and extend into the specimen ; within an enclave, the initial (elastic) distribution of χ has to be modified in accordance with (11). Allen and Southwell (1950) give a relaxational method for the computation of χ within a plastic enclave. The process is rather laborious in that the relaxation pattern differs from point to point, and strictly only applies to infinitesimal increments in χ. Much labour can be saved by the use of inexact patterns, *exact* account, of course, being kept of the residuals.

Fig. 4.



Values of the Mises-Hencky equivalent stress (taking 1000 as its maximum value) when the plastic enclave has spread right across the bar. Case $\frac{D}{d}=4$.

In all the examples considered here, a modified relaxation technique was adopted—inexact patterns being used to give the general trend of the χ distribution, the final values being obtained by a systematic trial-and-error method. The labour involved was not prohibitive. The determination of the growth of the plastic enclave is facilitated by the computational device adopted of proceeding from step to step by *lowering* the value of the maximum equivalent stress rather than by *increasing* the applied tension.

---

* Southwell has suggested the use of the word enclave to denote a region of plastic straining.

Four distinct stages in the growth of the plastic region were obtained. Fig. 4 records the values of the Mises-Hencky equivalent stress when the plastic enclave has spread until it extends right across the bar. Fig. 5 summarizes these four stages, and shows the growth of the enclave with increasing tension.

The same problem was considered for other values of $\frac{D}{d}$, the ratio of the total width of the bar to the width at the waist. The case $\frac{D}{d}=2$ exhibited the same general characteristics as the case $\frac{D}{d}=4$ already considered. The elastic solution was verified by photo-elastic methods, and excellent agreement was obtained in spite of the difficulty of obtaining a model with a "slit"—practical considerations necessitating the notch having a finite though small width. One difference in the case $\frac{D}{d}=2$ is the rapidity with which the plastic enclave spreads. Although the growth of the enclave exhibits the same characteristics as in the case $\frac{D}{d}=4$, only two stages in the growth were obtained, as compared with four in the previous case. Once plastic flow begins, the enclave spreads

Fig. 5.

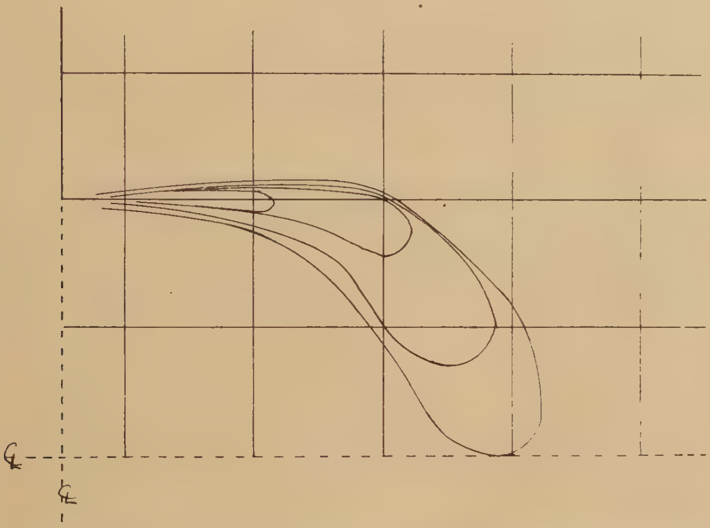
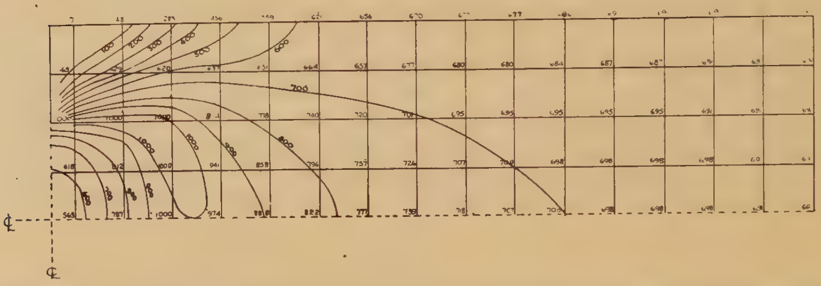


Diagram showing the progress of plastic straining—superposed plastic-elastic interfaces. Case $\frac{D}{d}=4$.

extremely rapidly and extends right across the bar for a very small increase in tension. Fig. 6 gives the values of the Mises-Hencky equivalent stress when the plastic enclave has spread right across the bar.

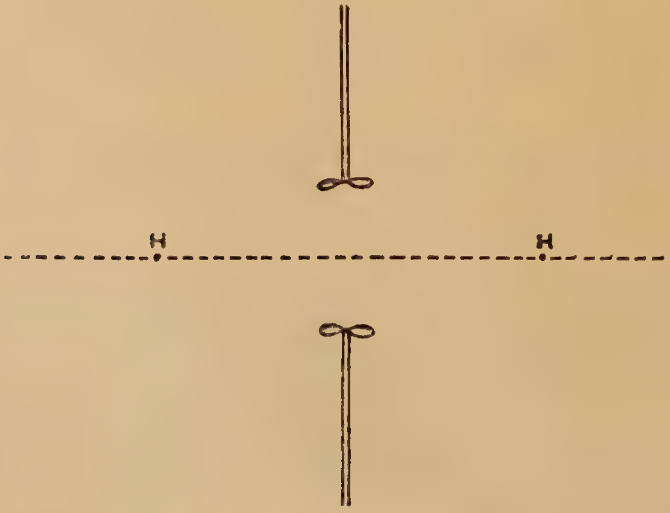
Fig. 6.



Values of the Mises-Hencky equivalent stress (taking 1000 as its maximum value) when the plastic enclave has spread right across the bar. Case $\frac{D}{d}=2$.

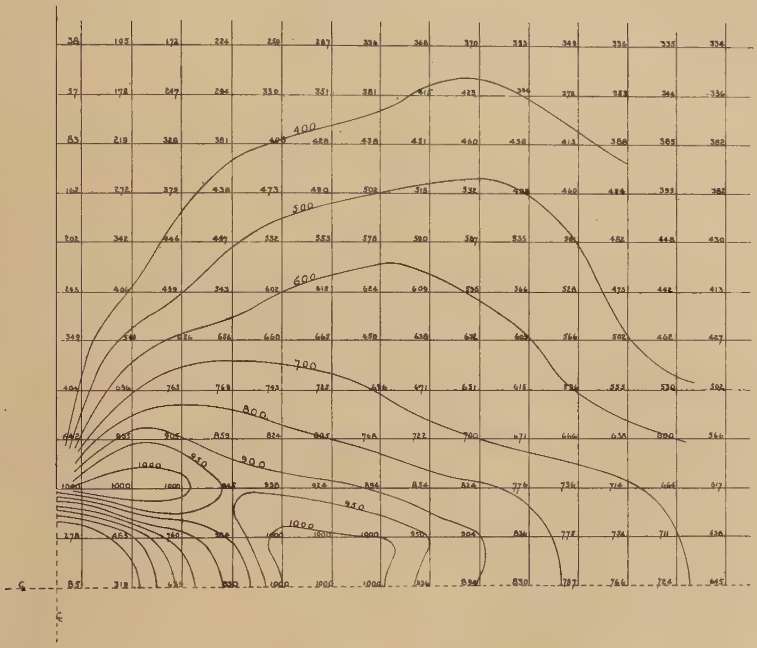
The case $\frac{D}{d}=8$ was an attempt to solve the problem for an infinite plate. There was a marked distinction in this case as compared with the other two cases. As plastic flow begins, an enclave, of the same shape as before, forms at the root of each notch and spreads into the plate; this

Fig. 7.



is accompanied a little later by two new enclaves starting from points H, H on the centre line near the waist (see fig. 7). This phenomenon was also found by Allen and Southwell (1950) in the case of notches which were semicircular. The enclaves extend extremely rapidly and, as in the case $\frac{D}{d}=2$, only one intermediate stage was obtained before the two enclaves coalesced. Fig. 8 gives the value of the Mises-Hencky equivalent stress for the most advanced stage of plastic straining obtained. As the ratio $\frac{D}{d}$ increases, so the labour of computation greatly increases, because the number of mesh points rapidly multiplies. The case $\frac{D}{d}=8$ should give a good indication of the results for an infinite plane, as there was little variation in the stress distribution a long way from the slit. Preliminary investigation for the case $\frac{D}{d}=10$ showed but slight variation from the case $\frac{D}{d}=8$.

Fig. 8.



Values of the Mises-Hencky equivalent stress (taking 1000 as its maximum value) for the final plastic stage. Case $\frac{D}{d}=8$.

§ 5. Summary of Results.

Hill (1949) considers the state of stress in the core of a notched bar under an applied load sufficient to cause plastic flow, the notch being of sufficient depth to ensure that the plastic region is localized in and around the core. If the curvature of the notch is greatest at the root, the yield limit is first attained at that point, and as the applied end load is further increased, Hill assumes that the plastic region will spread in some manner from the notch to the axis, and shows that this plastic region must be of a certain minimum extent when the bar first extends. The initial growth of the plastic region for the limiting case of a deep notch, viz. a slit, which has been considered here, is in a direction which could hardly have been forecast in advance, and is not towards the axis of the bar as suggested by Hill's diagrammatic figure. Nothing in the present paper, however, conflicts with his results and conclusions :— " So long as part of the core is not plastic, no displacement of the ends of the specimen is possible. Even when the plastic region has spread right across the core, it does not necessarily follow that deformation is possible, since the plastic material may be rigidly constrained by the adjacent bulk of non-plastic material ". From a consideration of the equations of Saint Venant governing the flow velocities in the plastic region, Hill shows that no extension of the specimen is possible until the plastic region has spread sufficiently far to include the entire slip lines OS, OS′ from the centre O to the free surface of the notch. At the moment when extension first occurs, the plastic boundary must touch OS, OS′, or must pass through either S and S′, or through O. Hill's surmise that the plastic boundary actually passes through S and S′ is confirmed by the results of this work :—

" Many writers, including some of the most recent, such as Sokolowsky (1946), and van Iterson (1947), have suggested slip-line fields for various problems quite arbitrarily, with no adequate discussion of their validity. In some cases elementary considerations are sufficient to prove them wrong, but in others only detailed numerical calculations can decide. In many problems the slip-line field in the region of plastically deforming material may be surmised without a full solution for the stresses in both elastic and plastic regions ". The work of Allen and Southwell (1950) and the results of the investigations presented here show that in many cases the actual plastic region develops in a direction which could hardly have been surmised in advance. Thus, although it may be easy to suggest apparently plausible slip-line fields for a number of problems, detailed numerical calculations constitute the only *certain* method of procedure. In a future paper the results will be given of the investigation of a sharp-ended bar under specified edge tractions and of the uniform compression of a block between two smooth, rigid plates which do not overlap it.

§ 6. ADDENDUM.

Since the above investigations were carried out, Hill has shown (1950) that in the plane deformation of an incompressible plastic-elastic material under a given loading-path the distribution of stress at any stage can only be found uniquely from the stress equations alone if the plastic-elastic boundary is not cut more than once by any slip-line (and if, further, unloading does not occur). His proof depends on the fact that the slip-lines in the plastic region are the characteristics of both the stress-rate equations and the velocity equations. It follows that the calculated plastic regions in this paper are, after a certain stage, possibly unreliable since theoretically an unlimited number could be found to satisfy the stress equations and boundary conditions. Uniqueness is obtained by choosing the plastic region so that a velocity solution can be found. The results given here, however, do exhibit correctly the growth of the plastic region in the initial stages, and it is not anticipated that the final results are much in error.

References.

Allen, D. N. de G., and Southwell, R. V., 1950, *Phil. Trans.* (to be published).
Christopherson, D. G., 1940, *Amer. J. Appl. Mech.*, **7,** A 1–4 (reprinted *Roy. Aero. Soc. J.*, **44,** 425–432).
Christopherson, D. G., and Southwell, R. V., 1938, *Proc. Roy. Soc.* A, **168,** 317–350.
Fox, L., and Southwell, R. V., 1945, *Phil. Trans.* A, **239,** 419–460.
Hill, R., 1949, *Quart. J. Mech. and Applied Math.*, **2,** 40–52.
Hill, R., 1950, *The Mathematical Theory of Plasticity* (to be published by the Clarendon Press, Oxford).
van Iterson, F. K. Th., 1947, *Plasticity in Engineering* (Blackie).
Prandtl, L., 1923, *Z. Angew. Math. Mech.*, **3,** No. 6, 442.
Shaw, F. S., 1942, *J. Inst. Eng. Australia*, **14,** 273–277.
Shaw, F. S., and Southwell, R. V., 1941, *Proc. Roy. Soc.* A, **178,** 1–17.
Sokolovsky, W. W., 1946, *Theory of Plasticity* (Izd. Ak. Nauk., Moscow).
Southwell, R. V., and Vaisey, G., 1946, *Phil. Trans.* A, **240,** 117–161.

XXIX. *On Tolman's Equation describing the Thermal Equilibrium in a Gravitating Sphere of Fluid.*

By H. A. Buchdahl,
Physics Department, University of Tasmania *.

[Received November 9, 1949.]

Abstract.

It is shown that by adopting a form of the line element slightly different from that chosen by Tolman a first order equation for the dependence of proper temperature on gravitational potential in a gravitating fluid sphere appears in place of Tolman's second order equation. The problem suggested by him concerning the meaning of a certain constant of integration therefore does not arise.

§1. Introduction.

Tolman (1934) has investigated the distribution of temperature in a gravitating sphere of fluid by invoking the thermodynamic conditions for static equilibrium. Assuming the form of the line element to be

$$ds^2 = -\exp\{\mu(r)\}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2)+\exp\{\nu(r)\}\,dt^2, \quad (1.1)$$

he arrives at a differential equation of the *second* order, a first integral of which is

$$\frac{d\log T_0}{dr} = -\frac{1}{2}\frac{d\nu}{dr} - \frac{B\exp\{-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu\}T_0}{r^2}, \quad (1.2)$$

where B is a constant of integration and T_0 is the absolute temperature as measured by a local observer at rest. In order to arrive at a solution of (1.2) in the form

$$T_0\exp\{\tfrac{1}{2}\nu\} = T_0\sqrt{(g_{44})} = \text{const.} \quad (1.3)$$

he sets B equal to zero " on physical grounds " and remarks that " it would of course be interesting also to investigate the possibility for solutions of physical interest in which the constant of integration was not taken to be zero ".

We shall show, however, that the appearance of the term involving B in (1.2) is spurious in the sense that by adopting a form of the line element slightly different from (1.1) there arises a *first* order equation which replaces Tolman's second order equation, and which has the integral (1.3). The question of having to assign a particular value to a constant of integration equivalent to B consequently never arises. It might be added that we are here not concerned with the question of the validity of Tolman's thermodynamics; and accordingly we make full use of his method of attacking the problem in hand.

---

* Communicated by the Author.

§2. DERIVATION OF THE EQUATION.

We adopt a spherically symmetric line element in the form

$$ds^2 = -\exp\{\lambda(r)\}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2) + \exp\{\nu(r)\}\,dt^2. \quad . \quad (2.1)$$

In the following a number preceded by the letter T refers to the equations of the work of Tolman quoted above; and his notation is adhered to throughout. Then, by T(95.13),

$$8\pi p_0 = \exp\{-\lambda\}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

$$8\pi \rho_{00} = \exp\{-\lambda\}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

Consequently

$$\begin{aligned}\delta E_0 &= \delta(4\pi\rho_{00}\exp\{\tfrac{1}{2}\lambda\}r^2\,dr)\\ &= (-2\pi\rho_{00}r^2 + \tfrac{1}{2})\exp\{\tfrac{1}{2}\lambda\}\delta\lambda\,dr + \tfrac{1}{2}r\exp\{-\tfrac{1}{2}\lambda\}\delta\lambda'\,dr,\end{aligned}$$

by (2.3). Also, by (2.3),

$$p_0\delta v_0 = p_0\delta(4\pi\exp\{\tfrac{1}{2}\lambda\}r^2\,dr) = 2\pi p_0 r^2\exp\{\tfrac{1}{2}\lambda\}\delta\lambda\,dr.$$

Hence

$$\begin{aligned}\delta E_0 + p_0\delta v_0 &= [-2\pi(\rho_{00} - p_0)r^2 + \tfrac{1}{2}]\exp\{\tfrac{1}{2}\lambda\}\delta\lambda\,dr + \tfrac{1}{2}r\exp\{-\tfrac{1}{2}\lambda\}\delta\lambda'\,dr\\ &= \{[\tfrac{1}{4}r(\nu' - \lambda') + \tfrac{1}{2}]\delta\lambda + \tfrac{1}{2}r\delta\lambda'\}\exp\{-\tfrac{1}{2}\lambda\}\,dr. \quad . \quad . \quad . \quad . \quad (2.4)\end{aligned}$$

Inserting (2.4) in the condition for thermodynamic equilibrium T(124.5) we have

$$\int\frac{\exp\{-\frac{1}{2}\lambda\}}{2T_0}\{[\tfrac{1}{2}r(\nu' - \lambda') + 1]\delta\lambda + r\delta\lambda'\}\,dr = 0 \quad . \quad . \quad . \quad (2.5)$$

for all arbitrary variations which vanish on the boundary of the region considered. Integrating the term containing $\delta\lambda'$ by parts, and noting that the integrated part vanishes by hypothesis, (2.5) becomes

$$\begin{aligned}&\int\left\{\frac{\exp\{-\frac{1}{2}\lambda\}}{2T_0}[\tfrac{1}{2}r(\nu' - \lambda') + 1] - \left(\frac{r\exp\{-\frac{1}{2}\lambda\}}{2T_0}\right)'\right\}\delta\lambda\,dr\\ &\qquad = \tfrac{1}{2}\int r\exp\{-\tfrac{1}{2}\lambda\}\left[\frac{\nu'}{2T_0} - \left(\frac{1}{T_0}\right)'\right]\delta\lambda\,dr = 0.\end{aligned}$$

Accordingly we must have

$$\left(\frac{1}{T_0}\right)' = \frac{\nu'}{2T_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.6)$$

This is the first order equation which replaces T(128.2); and its integral is again given by (1.3).

REFERENCE.

TOLMAN, R. C., 1934, *Relativity, Thermodynamics and Cosmology*, (Oxford University Press), Ch. IX., § 128, pp. 312–314.

XXX. *Scattering of Pseudoscalar Charged Mesons by Nucleons.*—II.

By E. Corinaldesi* and G. Field†,
Dublin Institute for Advanced Studies‡.

[Received January 4, 1950.]

Summary.

In § 1 the fourth- and sixth-order contributions to the scattering cross-section are evaluated for the non-relativistic region ($\omega \not> 3\mu$). The other section deals with damping effects.

§ 1. Sixth-order Corrections to the Scattering Cross-section.

We shall now investigate the same fourth-order transitions which were discussed in § 5 of Part I. (Corinaldesi and Field 1949). A more refined, still non-relativistic, treatment will be given here, for

$$\frac{P_i^2}{M^2},\quad \frac{p_i^2}{M^2},\quad \frac{P_f^2}{M^2},\quad \frac{p_f^2}{M^2} \ll 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

but with

$$P_i,\ p_i,\ P_f,\ p_f \sim \mu, \qquad (\omega \not> 3\mu). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

In order to simplify the calculation we shall choose as our reference system that in which the centre of gravity of the proton and meson is at rest. This will not affect the value which will be obtained for the cross-section. In the new reference system,

$$\left.\begin{aligned}\mathbf{P}_i &= -\mathbf{p}_i,\\ \mathbf{P}_f &= -\mathbf{p}_f.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

From the conservation of energy between the initial and final states, it follows that

$$|\mathbf{P}_i| = |\mathbf{p}_i| = |\mathbf{P}_f| = |\mathbf{p}_f|\ (= \mathrm{p, say}) \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

and

$$(\mathbf{p}_i \cdot \mathbf{p}_f) = \mathrm{p}^2 \cos\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where θ is the angle of scattering. We have also, with the notation of Part I.,

$$\omega_i = \omega_f = \omega$$

and

$$\Omega_i = \Omega_f = \Omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

---

* Now at Physics Department, University of Manchester.

† Now at Department of Mathematical Physics, University of Birmingham.

‡ Communicated by the Authors.

When conditions (1) are (2) are imposed, only eleven transitions are found to give contributions of the fourth-order which are not negligibly small. They are the same transitions that were discussed in Part I. All others have to be neglected, even for the higher energy range now under consideration.

Analogously to formula (49) of Part I. we have now

$$\mathrm{H}_{2,4}+\mathrm{H}_{4,2}=\frac{1}{8\Omega^2}\,\mathrm{Sp}\,\{\mathscr{H}_2(\Omega+\boldsymbol{\alpha}\,.\,\mathbf{P}_i+\mathrm{M}\beta)\mathscr{H}_4(\Omega+\boldsymbol{\alpha}\,.\,\mathbf{P}_f+\mathrm{M}\beta)$$
$$+\mathscr{H}_4(\Omega+\boldsymbol{\alpha}\,.\,\mathbf{P}_i+\mathrm{M}\beta)\mathscr{H}_2(\Omega+\boldsymbol{\alpha}\,.\,\mathbf{P}_f+\mathrm{M}\beta)\} \quad . \quad . \quad (7)$$

where

$$\mathscr{H}_2=-\frac{2\pi\mathrm{g}^2}{\mathrm{V}\omega}\left\{\frac{\omega-\Omega+\boldsymbol{\alpha}\,.\,\overline{\mathbf{P}}+\mathrm{M}\beta}{(\Omega-\omega)^2-\overline{\Omega}^2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\mathscr{H}_4=\frac{\pi^2\mathrm{g}^4}{2\mathrm{V}^2\omega}\sum_{\mathbf{p}'}\frac{1}{\omega(\mathbf{p}')\Omega(\mathbf{p}_1)\Omega(\mathbf{p}_2)\Omega(\mathbf{p}_3)(\mathrm{E}_i-\mathrm{E}_l)(\mathrm{E}_i-\mathrm{E}_m)(\mathrm{E}_i-\mathrm{E}_n)}\{\beta\gamma_5[\Omega(\mathbf{p}_1)$$
$$\mp\boldsymbol{\alpha}\,.\,\mathbf{p}_1\pm\mathrm{M}\beta]\beta\gamma_5[\Omega(\mathbf{p}_2)\pm\boldsymbol{\alpha}\,.\,\mathbf{p}_2\mp\mathrm{M}\beta]\beta\gamma_5[\Omega(\mathbf{p}_3)\mp\boldsymbol{\alpha}\,.\,\mathbf{p}_3\pm\mathrm{M}\beta]\beta\gamma_5\}. \quad . \quad . \quad . \quad (9)$$

In (9)

$$\left.\begin{aligned}\mathbf{p}_2&=\mathbf{p}',\\ \mathbf{p}_1&=\mathbf{p}'+\mathbf{p}_f\ \text{or}\ -\overline{\mathbf{P}},\\ \mathbf{p}_3&=\mathbf{p}'+\mathbf{p}_i\ \text{or}\ -\overline{\mathbf{P}}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Performing the integration over momentum space, and neglecting terms of second and higher orders in p/M as small compared with 1, we obtain

$$\mathrm{H}_{2,4}+\mathrm{H}_{4,2}=-\frac{3\pi}{4}\frac{\mathrm{g}^6}{\mathrm{V}^2\mathrm{M}^2\omega^2}\left(2+3\frac{\omega}{\mathrm{M}}-\frac{\mathrm{p}^2}{\mathrm{M}\omega}\cos\theta\right). \quad . \quad . \quad (11)$$

Similarly, $|\mathrm{H}_2|^2$ is evaluated from (8) and formula (16) of Part I.

$$|\mathrm{H}_2|^2=\frac{\pi^2\mathrm{g}^4}{\mathrm{V}^2\mathrm{M}^2\omega^2}\left(1+\frac{\omega}{\mathrm{M}}-\frac{\mathrm{p}^2}{\mathrm{M}\omega}\cos\theta\right). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The differential cross-section, $d\sigma$, is then given by

$$\begin{aligned}d\sigma&=d\sigma_4+d\sigma_6+\dots\\ &=\frac{\sin\theta\,d\theta\,d\phi}{(2\pi)^2}\,\omega^2\mathrm{V}^2[|\mathrm{H}_2|^2+\mathrm{H}_{2,4}+\mathrm{H}_{4,2}+\dots],\end{aligned} \quad . \quad . \quad . \quad (13)$$

in analogy with formula (55) of Part I. Thus the fourth-order term is

$$d\sigma_4=\frac{\mathrm{g}^4}{4\mathrm{M}^2}\left(1+\frac{\omega}{\mathrm{M}}-\frac{\mathrm{p}^2}{\mathrm{M}\omega}\cos\theta\right)\sin\theta\,d\theta\,d\phi \quad . \quad . \quad . \quad (14)$$

and the sixth-order term

$$d\sigma_6=-\frac{3}{16\pi\mathrm{M}^2}\,\mathrm{g}^6\left(2+3\frac{\omega}{\mathrm{M}}-\frac{\mathrm{p}^2}{\mathrm{M}\omega}\cos\theta\right)\sin\theta\,d\theta\,d\phi, \quad . \quad (15)$$

where $\sin\theta\,d\theta\,d\phi$ is the element of solid angle.

Integrating over θ and ϕ, we obtain for the fourth-order contribution to the *total* cross-section

$$\sigma_4 = \frac{\pi}{M^2} g^4 \left(1+\frac{\omega}{M}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14')$$

and for the sixth-order contribution

$$\sigma_6 = -\frac{3}{4M^2} g^6 \left(2+3\frac{\omega}{M}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15')$$

The ratio of the sixth- to the fourth-order terms is then given by

$$\left|\frac{\sigma_6}{\sigma_4}\right| = \frac{3}{4\pi} g^2 \left(2+\frac{\omega}{M}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This formula shows that, to the order of approximation used here, the ratio of the two contributions to the cross-section increases linearly with the energy. As for the convergence of the series $\sigma_4+\sigma_6+\ldots$, this depends, as was pointed out in Part I., on the choice of the numerical value of the coupling constant g.

At the present stage of the development of meson theory it is too early to say which of the values proposed by various authors is likely to be in agreement with experimental results—in so far as these already exist. In fact it is doubtful whether a pure pseudoscalar theory with a pseudoscalar coupling is likely to give a satisfactory account of the scattering process considered here. Several authors have adopted the value $g^2 \sim 36$, which would make the series divergent (Villars 1947, Dyson 1948, Luttinger 1948). Slotnick and Heitler (1949) have suggested a slightly lower value, $g^2 \sim 20$. On the other hand, Møller and Rosenfeld (1940), in their mixed theory, have chosen for the pseudoscalar coupling of pseudoscalar mesons a value $g^2 \sim 0{\cdot}1$, for which the series would converge. Recently, Bethe (1949) has been proposing yet another value, $g^2 \sim 4$.

Finally, we want to point out that the numerical value obtained for σ_6 depends on the way in which the renormalization of g is carried out. Since this point was not discussed in detail in Part I. of this paper, we shall give a short summary of the procedure here* :

After having carried out the mass and "charge" renormalization, we are left with $\sigma_6 = \sigma_6' + \sigma_6''$. Here, σ_6' is what remains of the previously divergent matrix elements, and σ_6'' is convergent. Now, σ_6' vanishes in the extreme non-relativistic case (E.N.R.) and even in the approximation (1) considered in this paper it does not give any contribution to the cross-section (15′), which is due only to σ_6''. In the case of the Compton effect, as treated by Corinaldesi and Jost (1948), the situation was different. There σ_6'', too, was zero in the E.N.R. case and this circumstance made it possible to say that the renormalization of the electric charge had been performed in such a way that $\lim \sigma_6 = 0$ for $p \rightarrow 0$. If we would apply the same criterion in the present calculation, the whole contribution to σ_6 which is constant for $p \rightarrow 0$ would have to be left out, and σ_6 would be much smaller than σ_4 for all energies under consideration. But we see no reason why we should apply this criterion to meson processes.

---

* We are grateful to Dr. M. P. Schafroth, Zürich, for asking us to clarify this point.

§ 2. DAMPING EFFECTS.

Another correction to the cross-section may be obtained if we consider the reaction of the emitted meson on the motion of the nucleon. Attempts have been made by several authors to take account of this damping reaction (Heitler and Peng 1942, Wilson 1941, Sokolov 1941, Gora 1943) and we shall briefly discuss its effects on the results obtained above.

The scattering cross-section may be written in the form

$$\sigma_D = 2\pi\rho(\omega)|U|^2, \quad \ldots\ldots\ldots \quad (17)$$

where U is a probability amplitude defined by the integral equation

$$U = K - i\pi K\rho U, \quad \ldots\ldots\ldots \quad (18)$$

U is a matrix element depending on the initial and final states, K is a similar matrix element given by $K = H_2 + H_4 + \ldots$ and $K\rho U$ is a matrix product, with summation over all final states. In order to obtain an idea of the order of magnitude of the damping effect it will not be necessary for us to solve the integral equation (18) rigorously. Substitution for U, in the damping term, of an average over angles will be sufficient for our purpose. We then obtain

$$U = \frac{K}{1 + i\pi\rho K}, \quad \ldots\ldots\ldots \quad (19)$$

so that the corrected cross-section σ_D is given by

$$\sigma_D = \frac{\sigma}{1 + \kappa^2}, \quad \ldots\ldots\ldots \quad (20)$$

where

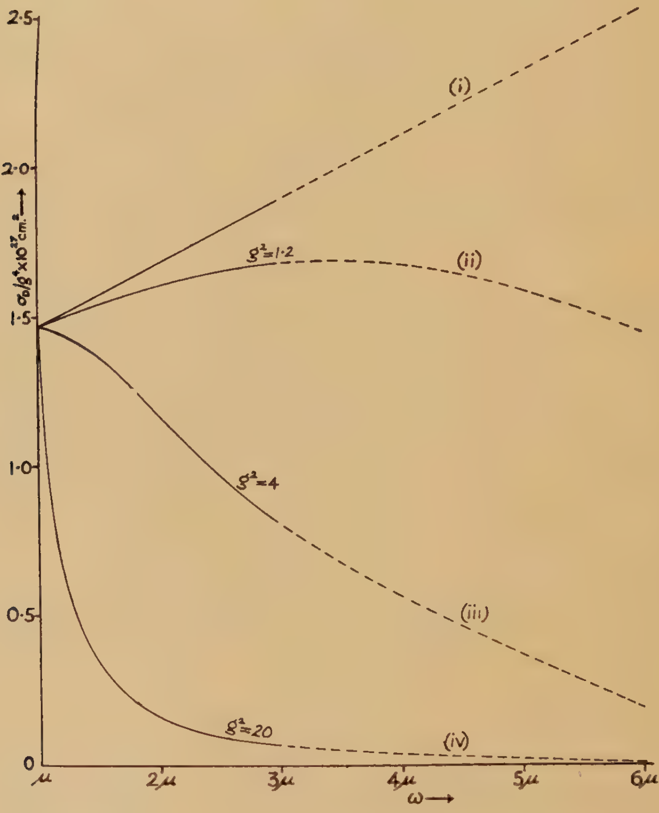
$$\sigma = 2\pi\rho(\omega)|K|^2 \quad \ldots\ldots\ldots \quad (21)$$

and

$$\kappa^2 = \pi^2\rho^2|K|^2. \quad \ldots\ldots\ldots \quad (22)$$

We shall now discuss the behaviour of σ_D for different values of g^2. For very high values of the coupling constant the series for $|K|^2$ diverges. The damping would still make the cross-section finite, namely $\sigma_D = \frac{2}{\pi\rho}$, but there is hardly any sense in including higher-order terms in the expressions for $|K|^2$ and κ^2, and this result would be very unreliable. In this case, we might rather assume that the first non-vanishing approximation of K comes closer to describing the real physical process than the inclusion of the rapidly increasing higher-order approximations would. Then we should have to put $|K|^2 = |H_2|^2$. At very low energies, the values of σ_D for $g^2=20$ are, of course, very much higher than those for $g^2=4$. With increasing energy, however, damping is much greater in the former case (since $\kappa^2 \propto g^4$) and at the upper limit of the range for which the formulæ are still valid (see (2)), cross-sections for both values of g^2 are of the same order of magnitude ($\sim 2\times 10^{-26}$ cm.^{2}, see fig. 1).

Fig. 1.

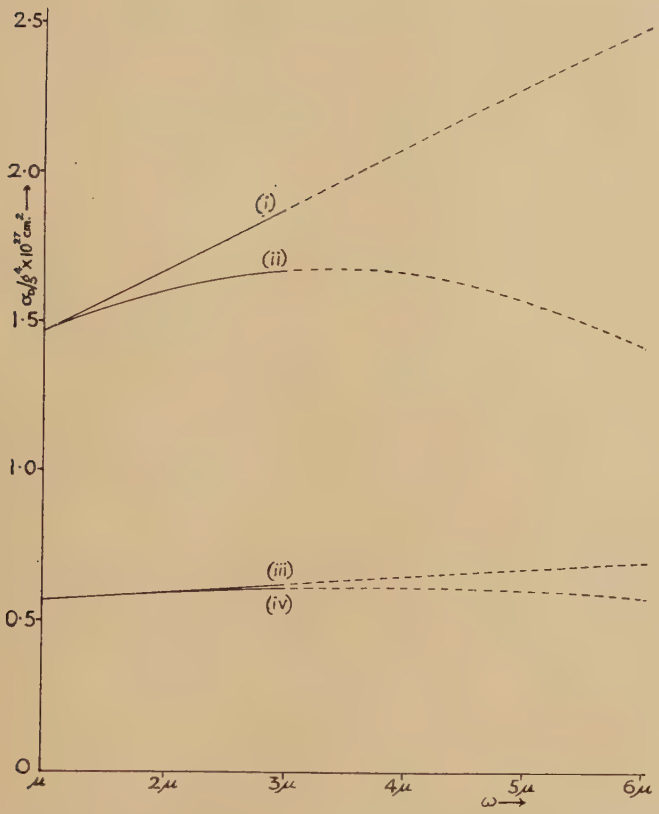


Energy dependence of damped cross-section, with $|K|^2=|H_2|^2$.
(i.) σ/g^4 (no damping),
(ii.), (iii.) and (iv.) σ_D/g^4 for $g^2=1{\cdot}2$, 4 and 20 respectively.

For lower values of g^2, *e. g.* $g^2=1{\cdot}2$, the damping effect is much smaller than in the previous cases. Moreover, it is now reasonable to include the sixth-order terms in the expression for $|K|^2$. Fig. 2 shows the damping effect for the two cases $|K|^2=|H_2|^2$ and $|K|^2=|H_2|^2+H_{2,4}+H_{4,2}$.

In conclusion, we would like to thank Professor W. Heitler for much valuable criticism.

Fig. 2.



Influence of sixth-order correction on cross-section, for $g^2=1{\cdot}2$.

(i.) σ/g^4 (no damping), with $|K|^2=|H_2|^2$.
(ii.) σ_D/g^4 with $|K|^2=|H_2|^2$.
(iii.) σ/g^4 (no damping), with $|K|^2=|H_2|^2+H_{2,4}+H_{4,2}$.
(iv.) σ_D/g^4 with $|K|^2=|H_2|^2+H_{2,4}+H_{4,2}$.

REFERENCES.

BETHE, H. A., 1949, *Phys. Rev.*, **76**, 191.
CORINALDESI, E., and FIELD, G., 1949, *Phil. Mag.*, **40**, 1159.
CORINALDESI, E., and JOST, R., 1948, *Helv. Phys. Acta*, **21**, 183.
DYSON, F. J., 1948, *Phys. Rev.*, **73**, 929.
GORA, E., 1943, *Zeits. f. Physik.*, **120**, 121.
HEITLER, W., and PENG, H. W., 1942, *Proc. Camb. Phil. Soc.*, **38**, 296.
LUTTINGER, J. M., 1948, *Helv. Phys. Acta*, **21**, 483.
MØLLER, C., and ROSENFELD, L., 1940, *D. Kgl. Danske Vid. Selsk.*, **17**, No. 8.
SLOTNICK, M., and HEITLER, W., 1949, *Phys. Rev.*, **75**, 1645.
SOKOLOV, A., 1941, *J. Phys. U.S.S.R.*, **5**, 231.
VILLARS, F., 1947, *Helv. Phys. Acta*, **20**, 476.
WILSON, A. H., 1941, *Proc. Camb. Phil. Soc.*, **37**, 301.

XXXI. *Influence of the Overlap Integral in Molecular-orbital Calculations for Alternant Hydrocarbons.*

By J. de Heer*,
Wheatstone Physics Department, King's College, London †.

[Received January 10, 1950].

Summary.

Some recent studies have indicated that in molecular-orbital calculations on alternant hydrocarbons the explicit inclusion of the overlap integral, S, between adjacent atomic orbitals makes no significant difference in the consistency of the results (Wheland 1942, Brooks 1941, Coulson and Chirgwin, in press). In the present paper a further investigation is reported on total π-electronic energies and the various polarizabilities of such hydrocarbons. In connection with a regularity observed by Wheland (1941), the resonance energies are discussed in relation to the total π-electronic energies; it appears that the latter are more consistent than the former. A correlation between the atom–atom–" self-polarizabilities " and the conjugation energies across single bonds leads us to define " conjugating powers " of hydrocarbon residues in complete concordance with the analogous definitions given in the simpler theory, neglecting overlap.

Introduction.

One of the most surprising features of the molecular-orbital method, if applied to *alternant hydrocarbons* (for a definition, see Coulson and Longuet-Higgins 1947 b), is the fact that the explicit inclusion of the overlap integral, S, between adjacent carbon $2p_z$-atomic orbitals makes no significant difference in the self-consistency of the calculations of several molecular properties. This was first shown by Wheland (1941) for the *resonance energies*, by comparing the results of calculations neglecting and including S respectively. In the same year Brooks (1941) found that computed values for the *diamagnetic anisotropy* of several aromatic ring systems are nearly unaffected by the inclusion of S. Recently, B. H. Chirgwin and C. A. Coulson (in press), in a more elaborate analysis, have proved that *mobile bond orders* and *π-electronic charge densities* remain completely unchanged, while they concluded that the various *self- and mutual polarizabilities* (introduced by Coulson and Longuet-Higgins 1947 a) are only slightly affected. Finally, unpublished calculations by C. A. Coulson indicate, that for a large number of polynuclear aromatic hydrocarbons the quotient β/γ, γ and β being the parameters in which the *total π-electronic energies* are expressed, when including and neglecting S respectively, is a fairly constant number. The last regularity is of the same character as—and obviously related to—the one concerning resonance energies.

---

* Now at the department of general and inorganic chemistry, the University of Amsterdam.

† Communicated by Professor C. A. Coulson, D.Sc.

Only in the case of bond orders and charge densities has a theoretical explanation been given. In all other cases the problem is strikingly similar: The quantities considered are the sum of certain contributions from the individual molecular orbitals or from pairs of them (*e. g.* the total π-electronic energy is twice the sum of the energies of the individual m.o.'s and the various polarizabilities are built up from contributions of pairs of one occupied and one unoccupied m.o.). Now it appears that while these separate contributions, being affected by some factor(s) containing S, may and in most cases do change appreciably and irregularly, their sum for some reason to be given either changes very little or is merely multiplied by a number, which is approximately the same for all molecules.

In the present investigation the explanation will be proved to lie in the fact that S, the value of which is usually taken to be 0·25, only occurs in the summed expressions as S^2, which is obviously a small quantity. First the total π-electronic energies and the resonance energies and then the polarizabilities will be discussed (the diamagnetic anisotropy is at present not tractable by our methods). It will be shown that total π-electronic energies are more consistent than resonance energies. Finally, conjugation energies, conjugating powers and their interrelation (Coulson and Longuet-Higgins 1948) will also be considered with the same inclusion of overlap. It turns out that here again certain conclusions drawn from the simpler theory, neglecting S, remain valid.

While the present analysis together with previous investigations referred to above, justifies us in neglecting overlap in molecular-orbital calculations on the ground-state of aromatic and olefinic hydrocarbons, this is no longer true in the case of heteromolecules. However, it still seems reasonable to assume that if few and small hetero-parameters occur in our equations, the inclusion of S will not disturb the consistency of the results essentially. If many and large such parameters are to be introduced the value of the entire simple molecular-orbital method as such becomes very doubtful.

1. The Total π-Electronic Energy.

If we neglect S, we can write for the energy of the r^{th} m.o., ϵ_r,

$$\epsilon_r = k_r\beta. \qquad (1)$$

Consequently the total π-electronic energy is given by

$$\mathscr{E} = \sum_{j=1}^{m} 2\epsilon_r, \qquad (2)$$

where the summation index, j, runs over the m occupied orbitals.

If we include S, taken to have the same numerical value for all pairs of adjacent atoms, we can write for the energy of the r^{th} m.o., E_r,

$$\mathrm{E}_r = m_r\gamma, \qquad \text{(3)}$$

where the new energy-parameter, γ, is given by

$$\gamma = \beta - \mathrm{S}\alpha, \qquad \text{(4)}$$

The relation between m_r and k_r is known to be (Wheland 1941, Coulson and Chirgwin, in press) :

$$m_r = \frac{k_r}{1+k_r\mathrm{S}}. \qquad \text{(5)}$$

The total π-electronic energy now becomes

$$\mathscr{E}_\mathrm{S} = \sum_{j=1}^{m} 2\mathrm{E}_r. \qquad \text{(6)}$$

Our notation implies that α, the Coulomb integral for the carbon centres, is taken as the zero level from which all energies are measured. In addition it should be remembered that S is a positive quantity, but β and γ are negative, while k and m are positive for all bonding orbitals.

From (6), (3) and (1) we derive

$$\mathscr{E}_\mathrm{S} = 2\gamma \sum_{j=1}^{m} \frac{\epsilon_r}{\beta + \mathrm{S}\epsilon_r}. \qquad \text{(7)}$$

This sum can be expressed by the complex integral :

$$\mathscr{E}_\mathrm{S} = \frac{\gamma}{\pi i}\int_{\Gamma}\left[\frac{z}{\beta+\mathrm{S}z}\cdot\frac{\Delta'(z)}{\Delta(z)}\right]dz, \qquad \text{(8)}$$

where the path of integration, Γ, is the imaginary axis from $-\infty i$ to $+\infty i$ and the infinite semi-circle to the left of the y-axis (fig. 1 ; compare Coulson 1940, Coulson and Longuet-Higgins 1947 a). $\Delta(z)$ is the secular-determinant :

$$\Delta(z) = \begin{vmatrix} -z & \beta_{12} & \beta_{13} \cdots\cdots \beta_{1n} \\ \beta_{21} & -z & \beta_{23} \cdots\cdots \beta_{2n} \\ \vdots & & \\ \vdots & & \\ \beta_{n1} & & -z \end{vmatrix}. \qquad \text{(9)}$$

Concerning equation (8) we first notice that the pole $z = -\beta/\mathrm{S}$ lies outside the contour. Further, if we compare this result with the alternative expression given by Coulson and Chirgwin (in press ; see equation (38) of their paper), then we observe that in (8) we have got rid of all quantities involving Coulson and Chirgwin's mixed tensor H_μ^ν and instead can deal with the same secular-determinant, (9), as in the simpler theory, neglecting S. This is achieved " at the cost of " introducing the complicating factor $1/(\beta+\mathrm{S}z)$ in the integrand.

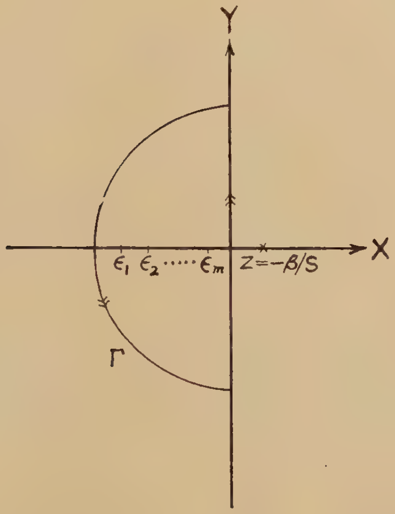
If we wish, we may replace (8) by

$$\mathscr{E}_{\mathrm{S}}=\frac{\gamma}{\pi i}\int_{F}\left[\frac{z}{\beta+\mathrm{S}z}\cdot\frac{\Delta'(z)}{\Delta(z)}-\frac{n}{\beta+\mathrm{S}z}\right]dz, \quad . \quad . \quad . \quad . \quad (10)$$

where n is the number of atoms in the molecule under consideration. Replacing (8) by (10) is correct, because

$$\int_{F}\frac{dz}{\beta+\mathrm{S}z}=0,$$

Fig. 1.



and it has the advantage that the contribution over the infinite semi-circle vanishes, since for large z,

$$\frac{z}{\beta+\mathrm{S}z}\cdot\frac{\Delta'(z)}{\Delta(z)}-\frac{n}{\beta+\mathrm{S}z}\cong\frac{1}{\mathrm{S}}\cdot\frac{nz^{n-1}}{z^{n}}-\frac{n}{\mathrm{S}z}=0.$$

Thus the integral (10) reduces to

$$\mathscr{E}_{\mathrm{S}}=\frac{\gamma}{\pi}\int_{-\infty}^{\infty}\left[\frac{iy}{\beta+\mathrm{S}iy}\cdot\frac{\Delta'(iy)}{\Delta(iy)}-\frac{n}{\beta+\mathrm{S}iy}\right]dy. \quad . \quad . \quad . \quad (11)$$

Multiplying numerator and denominator of the integrand by $(\beta-\mathrm{S}iy)$ yields

$$\mathscr{E}_{\mathrm{S}}=\frac{\gamma}{\pi}\int_{-\infty}^{\infty}\left[\frac{\beta iy+\mathrm{S}y^2}{\beta^2+\mathrm{S}^2y^2}\cdot\frac{\Delta'(iy)}{\Delta(iy)}+\frac{-n\beta+n\mathrm{S}iy}{\beta^2+\mathrm{S}^2y^2}\right]dy. \quad . \quad . \quad (12)$$

As $\frac{\Delta'(iy)}{\Delta(iy)}$ is an odd function of y, while of course $(\beta^2+S^2y^2)$ is even, the terms

$$\frac{Sy^2}{\beta^2+S^2y^2}\cdot\frac{\Delta'(iy)}{\Delta(iy)} \quad \text{and} \quad \frac{nSiy}{\beta^2+S^2y^2}$$

do not contribute to the integral. Hence equation (12) reduces to

$$\mathscr{E}_S=\frac{\gamma}{\pi}\int_{-\infty}^{\infty}\left[\frac{\beta iy}{\beta^2+S^2y^2}\cdot\frac{\Delta'(iy)}{\Delta(iy)}-\frac{n\beta}{\beta^2+S^2y^2}\right]dy,$$

which we finally write as

$$\mathscr{E}_S=\frac{\gamma}{\beta}\cdot\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\{1+S^2(y/\beta)^2\}}\cdot\left\{iy\frac{\Delta'(iy)}{\Delta(iy)}-n\right\}dy. \quad . \quad . \quad (13)$$

If we put S equal to zero, we obtain

$$\mathscr{E}=\frac{1}{\pi}\int_{-\infty}^{\infty}\left\{iy\frac{\Delta'(iy)}{\Delta(iy)}-n\right\}dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

which is equivalent to the result of the simpler theory (Coulson and Longuet-Higgins 1947 a), provided again that all energies are measured from α.

It will be noticed, that in (13) S only occurs as the small quantity S^2. Furthermore Coulson has shown (1940), that only the interval

$$0\leqslant|(y/\beta)|\leqslant 4$$

need be considered, in which interval the " correction-factor " varies from 1 to $\frac{1}{2}$. Finally the integrand in (14) usually reaches a rather steep maximum in the neighbourhood of $|y/\beta|=2$ (Coulson, private communication). If this maximum alone determined $\mathscr{E}_S$ in (13), then we should have

$$\mathscr{E}_S=0{\cdot}8\frac{\gamma}{\beta}\mathscr{E},$$

and we can expect the proper result to be only slightly different. In fact, as we shall see in the next paragraph, if we equate $\mathscr{E}$ and $\mathscr{E}_S$ as obtained by direct calculations, the result for a large number of molecules is a value for β/γ of about 0·70–0·75.

2. Resonance Energies and Total π-Electronic Energies ; the Quotient β/γ.

In this paragraph $(\beta/\gamma)_{\mathscr{E}}$ stands for the quotient β/γ which is found by equating $\mathscr{E}$ and $\mathscr{E}_S$, while $(\beta/\gamma)_R$ stands for the same quotient when obtained by equating the corresponding resonance energies, R and R_S. In Table I. values for $(\beta/\gamma)_{\mathscr{E}}$ and $(\beta/\gamma)_R$ for a number of molecules and free radicals have been collected. Most data for calculating these values have been published (see *e. g.* Wheland 1941, who used β and γ in the reverse sense) ; a few new computations were necessary.

As already mentioned in the introduction, Wheland (1941) pointed out that $(\beta/\gamma)_R$ is fairly constant. However, on closer inspection, it cannot be denied that rather large differences occur (up to 30 per cent from the mean value, 0·50 !), in particular between different types of molecule. On the other hand a glance at the last column of Table I. shows the values of $(\beta/\gamma)_{\mathscr{E}}$, which were not considered hitherto, to be far more consistent (0·73±3 per cent, if for the moment we consider ethylene as a separate case). This last fact is in accordance with the theoretical ideas developed in paragraph 1.

TABLE I.

Molecule	$(\beta/\gamma)_R$	$(\beta/\gamma)_{\mathscr{E}}$
Benzene	0·54	0·73
Biphenyl	0·51	0·72
Naphthalene	0·51	0·72
Anthracene	0·49	0·72
Phenanthrene	0·50	0·72
Pyrene	0·49	0·71
1, 3, 5-triphenylbenzene	0·50	0·72
Fulvene	0·44	0·73 *
Azulene	0·48	0·72 *
Styrene	0·50	0·73
Stilbene	0·50	0·72
1, 1-diphenyl-ethylene	0·50	0·72
1-phenyl-butadiene	0·48	0·73
2-phenyl-butadiene	0·48	0·73
Ethylene	—	0·80
Butadiene	0·36	0·75
Hexatriene	0·39	0·74
Octatetræne	0·40	0·74
Decapentæne	0·41	0·73
Dodecahexæne	0·42	0·73
Benzyl	0·56	0·72
α-Naphthylmethyl	0·65	0·72
β-Naphthylmethyl	0·64	0·72
m-biphenylmethyl	0·56	0·72
p-biphenylmethyl	0·58	0·72
diphenylmethyl	0·53	0·72
triphenylmethyl	0·51	0·72
Allyl	0·63	0·75

---

* These hydrocarbons contain odd-membered rings, and are therefore not alternant.

It can easily be shown, that to a first approximation, a constant value of $(\beta/\gamma)_{\mathscr{E}}$ of about 0·73 need not give rise at all to a similar uniformity amongst $(\beta/\gamma)_{\mathrm{R}}$-values. Let us take the case of an alternant hydrocarbon with an even number, n, of carbon atoms. Let us put

$$\mathscr{E}=\mathrm{K}\beta.$$

Then according to our assumption above,

$$\mathscr{E}_{\mathrm{S}}\cong 0{\cdot}73\mathrm{K}\gamma.$$

To obtain R and $\mathrm{R_S}$ in the usual way, we have to subtract from these total π-electronic energies $n/2$ times the total π-electronic energy of an ethylene molecule (2β and $1{\cdot}6\gamma$ respectively). Hence

$$\mathrm{R}=(\mathrm{K}-n)\beta,$$

$$\mathrm{R_S}\cong(0{\cdot}73\mathrm{K}-0{\cdot}8n)\gamma.$$

Equating R and $\mathrm{R_S}$ gives us the result

$$(\beta/\gamma)_{\mathrm{R}}\cong 0{\cdot}73-0{\cdot}07\,.\,\frac{1}{\left\{\frac{\mathrm{K}}{n}-1\right\}}. \qquad (15)$$

As K/n is greater than 1, $(\beta/\gamma)_{\mathrm{R}}$ can be expected to be smaller than 0·73. This indeed is found by direct calculations (Table I.). Furthermore, as K/n may show rather large fluctuations, equation (15) cannot be expected to yield very similar values of $(\beta/\gamma)_{\mathrm{R}}$ for all types of molecule; *e. g.* comparing benzene with the open chain hydrocarbon with the same n, hexatriene, shows us very clearly what happens. For these molecules $(\beta/\gamma)_{\mathscr{E}}$ is 0·73 and 0·74 respectively. For benzene K/n=1·33, for hexatriene K/n=1·16. As a result we find a large difference in $(\beta/\gamma)_{\mathrm{R}}$ (the values are 0·54 and 0·39 respectively). In this way, a detailed analysis of K/n might give us, for some type of molecule or radical, a more complete explanation of the $(\beta/\gamma)_{\mathrm{R}}$ values as obtained by direct calculations, but this seems of minor importance to us.

Obviously the cause of all these difficulties lies in the fact that ethylene, which plays such an important part in the calculation of the resonance energies, stands alone in having an exceptionally high value of $(\beta/\gamma)_{\mathscr{E}}$, namely 0·8. There is no way of getting round this difficulty and the important point arising from this discussion is that resonance energies calculated with the inclusion of overlap for different types of molecule need not be reflected very accurately by the values obtained by means of the simpler theory, neglecting S. Total π-electronic energies appear to be far more consistent and should be given preference if such is possible within the scope of a particular investigation.

3. POLARIZABILITIES.

The definition of the various polarizabilities and several formulæ to calculate them have been given by Coulson and Longuet-Higgins (1947 a) within the framework of the simpler theory, neglecting overlap. Coulson and Chirgwin (in press) have recently shown how the inclusion of S necessitates a change in the formulæ, while some numerical calculations indicate that apart from a factor β/γ there is extremely little difference between the two sets of figures thus obtainable.

The short analysis to follow will be given for the specific case of the atom–atom "self-polarizabilities", but it holds equally well for all the others. When S is neglected we shall use the customary notation $\pi_{r,\,r}$ for the self-polarizability of carbon centre r in a particular alternant hydrocarbon; when S is taken into account, we shall write $(\pi_{r,\,r})_S$ instead. In Table II. a few calculated values have been collected to illustrate the similarity between the two cases, for whose existence we must find some explanation.

TABLE II.

Hydrocarbon residue	$\beta\,.\,\pi_{r,r}$	$\gamma\,.\,(\pi_{r,r})_S$
Phenyl	0·398	0·359
2-Butadienyl	0·402	0·377
Vinyl	0·500	0·469
1-Butadienyl	0·626	0·601
1-Hexatrienyl	0·685	0·661

Without including S, we have (Coulson and Longuet-Higgins, 1947 a):

$$\pi_{r,\,r}=4\sum_{j=1}^{m}\sum_{h=m+1}^{n}\frac{c_{rj}^2\,.\,c_{rh}^2}{\epsilon_j-\epsilon_h}, \qquad \ldots\ldots\ldots \quad (16)$$

where ϵ_j stands for the energy of the j^{th} molecular orbital, as before, and c_{rj} is the coefficient of the r^{th} atomic orbital in this m.o. The summation indices j and h refer to the occupied and unoccupied molecular orbitals respectively. (In (16) the index h is used instead of the more customary index k; this is done to prevent confusion with the k, introduced in equation (1).)

When including S, equation (16) has to be replaced by (Coulson and Chirgwin, in press)

$$(\pi_{r,\,r})_S=\frac{4\beta}{\gamma}\sum_{j=1}^{m}\sum_{h=m+1}^{n}\frac{c_{rj}^2\,.\,c_{rh}^2}{\lambda_j^2\lambda_h^2(\epsilon_j-\epsilon_h)}, \qquad \ldots\ldots \quad (17)$$

where

$$\left.\begin{aligned}\lambda_j^2&=1/(1+k_jS)\\ \lambda_h^2&=1/(1+k_hS)\end{aligned}\right\} \qquad \ldots\ldots\ldots \quad (18)$$

and where k_j and k_h are related to ϵ_j and ϵ_h according to equation (1). Substituting (18) in (17) yields

$$(\pi_{r,\,r})_S=\frac{4\beta}{\gamma}\sum_{j=1}^{m}\sum_{h=m+1}^{n}[1+(k_j+k_h)S+k_jk_hS^2]\frac{c_{rj}^2\,.\,c_{rh}^2}{\epsilon_j-\epsilon_h}.$$

The double sum in the r.h.s. consists of three terms of which the first one is evidently $\beta/\gamma \, . \, \pi_{r,\,r}$. The second term, the one involving S to the first degree, vanishes owing to the symmetric distribution of occupied and unoccupied m.o.'s around the zero-level (Coulson and Longuet-Higgins, 1947 b). So finally we obtain

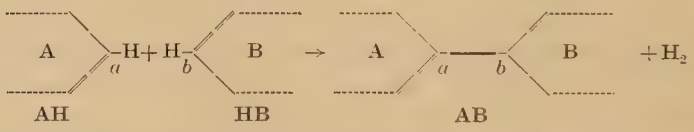
$$\gamma \, . \, (\pi_{r,\,r})_S = \beta \, . \, \pi_{r,\,r} + 4S^2 \sum_{j=1}^{m} \sum_{h=m+1}^{n} \frac{k_j k_h}{k_j - k_h} \, . \, c_{rj}^2 c_{rh}^2 . \quad . \quad . \quad (19)$$

So the difference between $\gamma \, . \, (\pi_{r,\,r})_S$ and $\beta \, . \, \pi_{r,\,r}$ will always be small as it only involves S^2. Furthermore this " correction-term " will always be negative (compare Table II.) as all terms in the sum in the r.h.s. of (19) are negative. If we wish, we can use equation (19) for the numerical calculation of $(\pi_{r,\,r})_S$ if $\pi_{r,\,r}$ is known. *E. g.* for ethylene (one occupied and one unoccupied m.o.), with S taken to be 0·25 as usual, we obtain immediately

$$\gamma(\pi_{r,\,r})_S = 0{\cdot}500 + 4 \, . \, \frac{1}{16} \, . \, \frac{1 \, . \, (-1)}{1-(-1)} \, . \, \frac{1}{2} \, . \, \frac{1}{2} = 0{\cdot}500 - \frac{1}{32} = 0{\cdot}469,$$

a result which, of course, is also obtained by the direct application of equations (17) and (18).

4. CONJUGATION ENERGY AND CONJUGATING POWERS.



The " Conjugation energy across a single bond " was defined by Coulson and Longuet-Higgins (1948) as follows (see diagram above): " Let us connect two alternant hydrocarbons, AH and HB, by eliminating two hydrogen atoms at carbon atoms *a* of AH and *b* of HB respectively and subsequently forming the bond *a–b*. Then the conjugation energy across this bond, R_{a-b}, is defined by (see diagram above)

$$-R_{ab} = \mathscr{E}(AB) - \mathscr{E}(AH) - \mathscr{E}(HB), \quad . \quad . \quad . \quad . \quad (20)$$

where, in accordance with our previous notation, the $\mathscr{E}$'s refer to the total π-electronic energies of the species indicated in brackets ".

Coulson and Longuet-Higgins (1948) show that the following relation holds in good approximation :

$$-R_{ab}/\beta = \text{constant} \times \beta \, . \, \sqrt{(\pi_{a,a} \, . \, \pi_{b,b})}, \quad . \quad . \quad . \quad . \quad (21)$$

where $\pi_{a,\,a}$ and $\pi_{b,\,b}$ are the self-polarizabilities of the carbon atoms *a* and *b* in the systems AH and HB respectively. On account of this result the self-polarizability is considered as the " conjugating power " of a hydrocarbon residue.

If we include the overlap integral S in our calculations, we must obviously replace the defining equation (20) by

$$-(R_{ab})_S = \mathscr{E}_S(AB) - \mathscr{E}_S(AH) - \mathscr{E}_S(HB). \quad . \quad . \quad . \quad (20a)$$

Fig. 2 (*a*).

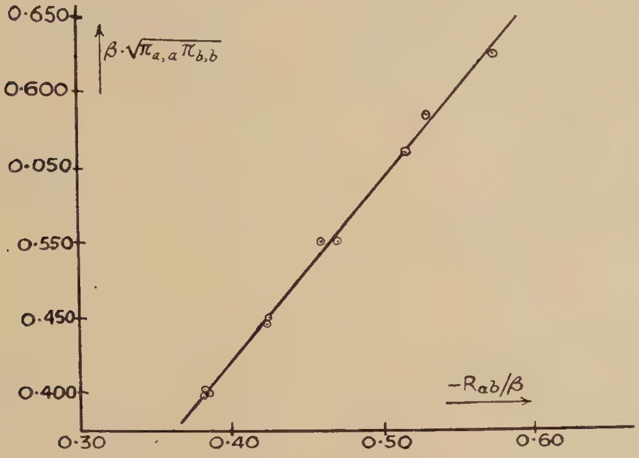
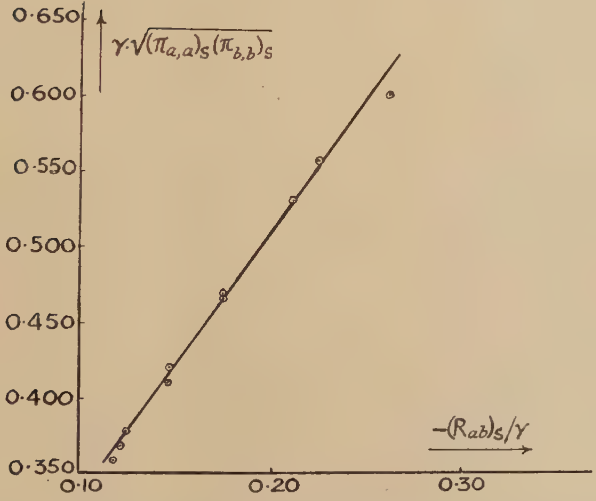


Fig. 2 (*b*).



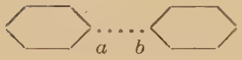
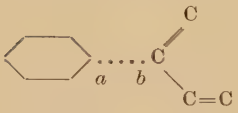
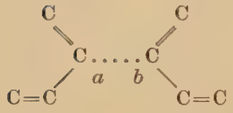
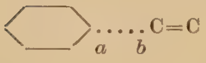
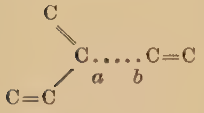
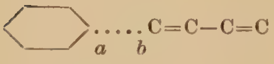
Then it is a natural question to ask whether a relationship still holds equivalent to the one embodied in (21), namely

$$-(R_{ab})_S/\gamma = \text{constant} \times \gamma \, . \, \sqrt{[(\pi_{a,a})_S \cdot (\pi_{b,b})_S]}. \quad . \quad . \quad (21a)$$

If so, then $(\pi_{a,a})_S$ and $(\pi_{b,b})_S$ act as conjugating powers, just as $\pi_{a,a}$ and $\pi_{b,b}$ did before.

A theoretical treatment is not possible here. It must be remembered that equation (21) was never derived rigorously, but even the equivalent to the approximate treatment, as given by Coulson and Longuet-Higgins (1948) cannot be given here, for the integral expressions for $\mathscr{E}_S$ and the $(\pi_{r,r})_S$'s become too involved for this purpose. So we can only check equation (21 a) by direct numerical calculations. In fig. 2 (*a*) we have therefore plotted computed values of $-R_{ab}/\beta$ against $\beta . \sqrt{[\pi_{a,a} \cdot \pi_{b,b}]}$ and in fig. 2 (*b*) $-(R_{ab})_S/\gamma$ against $\gamma . \sqrt{[(\pi_{a,a})_S \cdot (\pi_{b,b})_S]}$. The numerical data concerned have been collected in Table III.

TABLE III.

	$-R_{ab}/\beta$	$\beta.\sqrt{(\pi_{a,a} . \pi_{b,b})}$	$-(R_{ab})_S/\gamma$	$\gamma.\sqrt{[(\pi_{a,a})_S . (\pi_{b,b})_S]}$
	0·383	0·398	0·118	0·359
	0·386	0·400	0·121	0·368
	0·388	0·402	0·124	0·377
	0·424	0·446	0·146	0·410
	0·427	0·448	0·147	0·421
	0·460	0·499	0·174	0·464
C=C.....C=C *a* *b*	0·472	0·500	0·175	0·469
C=C–C=C.....C=C *a* *b*	0·516	0·559	0·212	0·531
C=C–C=C–C=C.....C=C *a* *b*	0·530	0·586	0·225	0·557
C=C–C=C.....C=C–C=C *a* *b*	0·574	0·626	0·263	0·601

A comparison of figs. 2 (*a*) and 2 (*b*) shows very clearly that equation (21 a) holds in as good an approximation as equation (21). This is not a complete surprise as we have shown in previous paragraphs that the values of $\mathscr{E}$ and $\pi_{r,r}$ are, in general, consistent with $\mathscr{E}_S$ and $(\pi_{r,r})_S$. These quantities completely determine the conjugation energies and the conjugating powers. Thus here again the explicit inclusion of the overlap integral does not upset the qualitative and semi-quantitative picture as given by the simpler theory. This result is particularly gratifying since some recent investigations show (Coulson, Evans and de Heer, to be published) that the conjugating powers in alternant hydrocarbons are also of significance in determining certain properties of the π-electronic system of substituted hydrocarbons.

Note added in proof.—Professor J. A. A. Ketelaar and Mr. J. van Dranen have drawn the attention of the writer to the following point: It is not clear from the manuscript that the complicated analysis of paragraph 1 is superfluous if our object is simply to conclude that total π-electronic energies are more consistent with respect to the inclusion of overlap than resonance energies. They point out that we have, by definition (notation as in paragraph 2):

$$\frac{\mathscr{E}_S}{\mathscr{E}}=(\beta/\gamma)_{\mathscr{E}}=\frac{0{\cdot}8n+R_S}{1{\cdot}0n+R}.$$

Combining the two facts that:

(i) R_S is small compared with $0{\cdot}8n$ and R is small compared with $1{\cdot}0n$,

(ii) Wheland (1941) has shown by direct calculations that R_S/R values are *fairly* constant,

it is more or less trivial that $\mathscr{E}_S/\mathscr{E}$ and $(\beta/\gamma)_{\mathscr{E}}$ are better constants than R_S/R and $(\beta/\gamma)_R$ respectively. It is also immediately clear from the above equation that the $(\beta/\gamma)_{\mathscr{E}}$ will in general be slightly smaller than $0{\cdot}8$ and can at most be equal to this value, namely if $R_S=R=0$ (ethylene).

In our investigation we have proved theoretically that the consistency of $\mathscr{E}_S/\mathscr{E}$—and thus of $(\beta/\gamma)_{\mathscr{E}}$—values and their magnitude ($\pm 0{\cdot}8$) can be expected *a priori*. We have subsequently shown, that if we take this result as a starting point and then proceed to define resonance energies in the usual way, it becomes immediately evident that $(\beta/\gamma)_R$-values will in general not show the same consistency. This second step is entirely equivalent to the analysis, given by Ketelaar and van Dranen (see above), but in the reverse sense.

ACKNOWLEDGMENT.

The writer wishes to thank Professor C. A. Coulson for many helpful suggestions and discussions. He also wishes to express his gratitude to Professor C. A. Coulson and Mr. B. H. Chirgwin for enabling him to read their manuscript prior to publication.

REFERENCES.

HARVEY BROOKS, 1941, *J. Chem. Phys.*, **9**, 463.
COULSON, C. A., 1940, *Proc. Camb. Phil. Soc.*, **36**, 201.
COULSON, C. A. and CHIRGWIN, B. H., *Proc. Roy. Soc.* (in press).
COULSON, C. A. and LONGUET-HIGGINS, H. C., 1947 a, *Proc. Roy. Soc.* A, **191**, 39; *Ibid.*, 1947 b, A, **192**, 16; *Ibid.*, 1948, A, **195**, 188.
WHELAND, G. W., 1941, *Journ. Am. Chem. Soc.*, **63**, 463.

XXXII. *The Estimation of Velocities and Local Mach Numbers in the Flow of a Compressible Fluid through Divergent Nozzles, under Adiabatic and Isothermal Conditions.*

By R. K. TEMPEST *.

[Received January 25, 1950.]

SUMMARY.

In the present paper, a theory is developed giving the approximate velocity at any point in an inviscid compressible fluid flowing through a divergent nozzle. The method of calculation assumes a radial pattern for the stream lines, as suggested by the theoretical and experimental work on flows in diffusers carried out by Gourzhienko (1939). The velocity field is extended by successive applications of the known solution for radial flow from a point source. The formulæ obtained in the present paper are applicable to either subsonic or supersonic flows, they may be applied over any length of tube ; and from them the variation of the local speed of sound and Mach number may be estimated.

Numerical examples are given in tables to demonstrate the very large increases in local velocity and Mach number in supersonic flows associated with comparatively small variation of the expansion ratio of the nozzles.

A comparison with the results of one-dimensional theory has been made in the case where the velocity is approximately uniform across sections perpendicular to the axis of the nozzle. It is pointed out that one-dimensional theory may not be applied to a flow in which a considerable change of velocity occurs across a section ; this type of flow is described by the theory of this paper.

§ 1. INTRODUCTION.

WE consider the flow of a frictionless, compressible fluid in a nozzle with axial symmetry. In general discussions of the flow of compressible fluids in tubes, linearized equations of motion are used to provide a solution valid in some small length of tube. The assumption is made that the Mach number and the local speed of sound are constant over this length, or, more accurately, variations in these quantities are assumed to be comparatively small and are neglected in the coefficients of terms that are themselves small. Numerical evaluation of the velocity at any point is cumbersome owing to the complexity of the exact differential equation for the potential governing the flow. Under certain circumstances, however, these equations may be simplified to give a formula for the velocity at any point in very simple form. The theory given below has the advantages that it may be applied over any length of tube and that the variation of the speed of sound and local Mach number are taken into consideration and estimated.

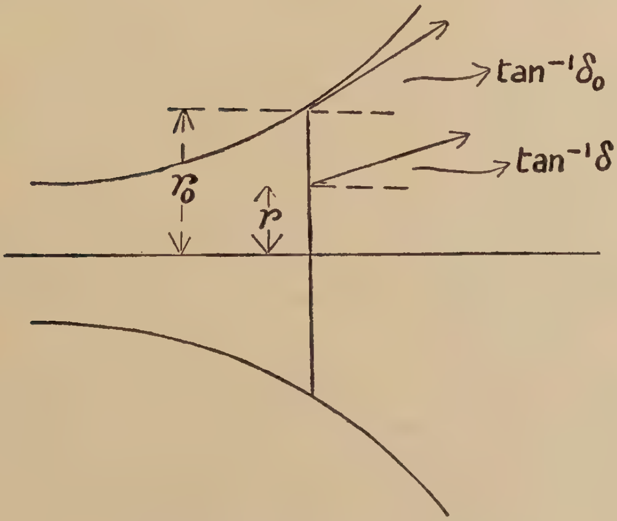
---

* Communicated by the Author.

We shall first investigate the flow in a cone which is purely radial from a source at the vertex and in which the slope of a stream line through any point in the field is proportional to the displacement of the point from the central stream line. This state of affairs gave Gourzhienko good experimental agreement in the case of flows which are not completely conical and is considered to be a basic characteristic of flows in nozzles. We shall assume that in a small element of the nozzle formed by two cross-sections at right angles to the axis, this gradation of stream lines exists. Thus the flow in such a section will be described by a function which in some respects plays the part of a source placed at the vertex of the truncated cone formed by the two sections. If the slope of the nozzle at a particular section with radius r_0 is δ_0, then we assume that the slope of the stream line through the circle of radius r in that section is (see fig. 1)

$$\delta=\left(\frac{r}{r_0}\right)\delta_0. \qquad (1)$$

Fig. 1.



Variation of direction of stream lines.

In practice, the use of diffusers has been confined to those of conical or almost conical shape, and no experimental evidence on the slope of the stream lines in a general divergent flow is available. Gourzhienko (1939) found that in the flow through a cylindrical-conical-cylindrical pipe the “ assumption of radiality ” was justified. The flows considered in this paper are confined to pipes which have none of the discontinuities in slope of the type mentioned above, and we should expect that the assumption would be even more justified in these cases.

§ 2. NOTATION.

We take the axis of the tube as the z-axis ; the coordinate defining distance perpendicular to the axis is r, and corresponding spherical coordinates are (R, θ). In irrotational flows, the equation satisfied by the potential is

$$\phi_{rr}+\frac{1}{r}\phi_r+\phi_{zz}=\frac{1}{a^2}(\phi_r^2\phi_{rr}+2\phi_r\phi_z\phi_{rz}+\phi_z^2\phi_{zz}) \quad . \; . \; . \; . \quad (2.1)$$

or

$$\phi_{\mathrm{RR}}+\frac{2}{\mathrm{R}}\phi_{\mathrm{R}}+\frac{1}{\mathrm{R}^2}\phi_{\theta\theta}+\frac{\cot\theta}{\mathrm{R}^2}\phi_\theta$$
$$=\frac{1}{a^2}\left(\phi_{\mathrm{R}}^2\phi_{\mathrm{RR}}+\frac{2}{\mathrm{R}^2}\phi_{\mathrm{R}}\phi_\theta\phi_{\mathrm{R}\theta}+\frac{\phi_\theta^2}{\mathrm{R}^4}\phi_{\theta\theta}-\frac{\phi_{\mathrm{R}}\phi_\theta^2}{\mathrm{R}^3}\right), \quad . \quad (2.2)$$

where a is the local speed of sound and is a function of velocity if the pressure is a function of density only. Equations (2.1) and (2.2) are forms of the general equation for the potential in the appropriate coordinates (Durand 1934). If an adiabatic law of the form $p=k\rho^\gamma$ applies, then

$$a^2=\frac{dp}{d\rho}=k\gamma\rho^{\gamma-1},$$

and Bernouilli's equation assumes the form

$$a^2=\frac{\gamma-1}{2}(\mathrm{Q}^2-q^2); \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2.3)$$

where q is the velocity at Mach number q/a, and Q is the speed of flow into vacuum, which under adiabatic conditions, is the greatest speed attainable. Under isothermal conditions, the speed of sound is constant and equal to a_0 ; Bernouilli's equation is then

$$a_0^2 \ln \rho+\tfrac{1}{2}q^2=\text{constant}. \quad . \; . \; . \; . \; . \; . \; . \quad (2.4)$$

In this case there is no limit to the speed.

§ 3. THE FUNDAMENTAL SUPERSONIC FLOW IN A DIVERGENT CONE.

The solution is exact and the flow is irrotational. The gas state is constant on spheres centred at the source, and the velocity increases along radial lines from this point. The formula relating velocity to distance R is given by the equation of continuity which assumes the form

$$\rho \mathrm{R}^2 q=\text{constant}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3.1)$$

From Bernouilli's equation (2.3), we immediately obtain

$$(\mathrm{Q}^2-q^2)^{1/\gamma-1} q \mathrm{R}^2=\text{constant}. \quad . \; . \; . \; . \; . \; . \quad (3.2)$$

Let us now suppose that the velocity at a point denoted by R is u (see fig. 2).

It immediately follows from (3.2) that

$$\frac{u_0}{u}\left(\frac{Q^2-u_0^2}{Q^2-u^2}\right)^{1/\gamma-1}=\left(\frac{R}{R_0}\right)^2, \quad \ldots \ldots \quad (3.3)$$

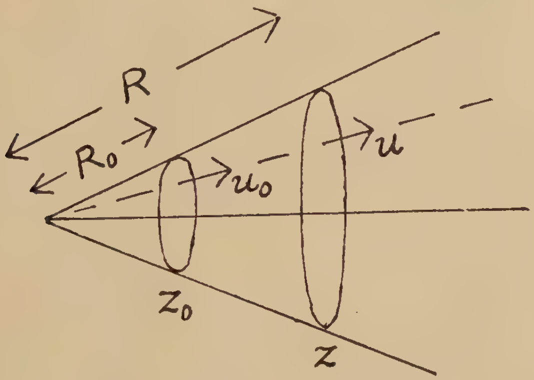
which gives the outlet velocity in terms of the inlet velocity, the speed of flow into vacuum, and the dimensions of the tube.

It should be noted that equation (3.3) holds in both supersonic and subsonic cases ; it follows from Bernouilli's equation that in the supersonic case, Q^2 is certainly less than $(\gamma+1/\gamma-1)u_0^2$, and for very high speeds will be considerably less than this. In subsonic flows Q^2 is greater than $(\gamma+1/\gamma-1)u_0^2$.

§ 4. Flow in a Divergent Nozzle.

It is assumed that the equation of the nozzle boundary is $r=f(z)$. Generally, the nozzle will be of the form indicated in fig. 3.

Fig. 2.



Velocity distribution in a conical flow.

The initial velocity on a representative stream line is u_0 ; the velocity at the point where this stream line intersects a z-constant section is u. It is supposed that the distribution of velocity u_0 across the inlet is known. By use of fig. 1, and a knowledge of u as a function of u_0, we may find the velocity at any point on any streamline. If u_0 is constant, the velocity will be constant across any section perpendicular to the axis, and we may estimate the section at which a particular velocity or a particular Mach number is attained.

We assume that the flow between the two sections z and $z+\delta z$ appear to diverge from the point on the axis with coordinate $z-[f(z)/df/dz]$, or that the slope of the stream lines is proportional to distance from the axis. This assumption in effect determines the equations of the stream lines, since from (1.1) we may write the differential equation of the family as

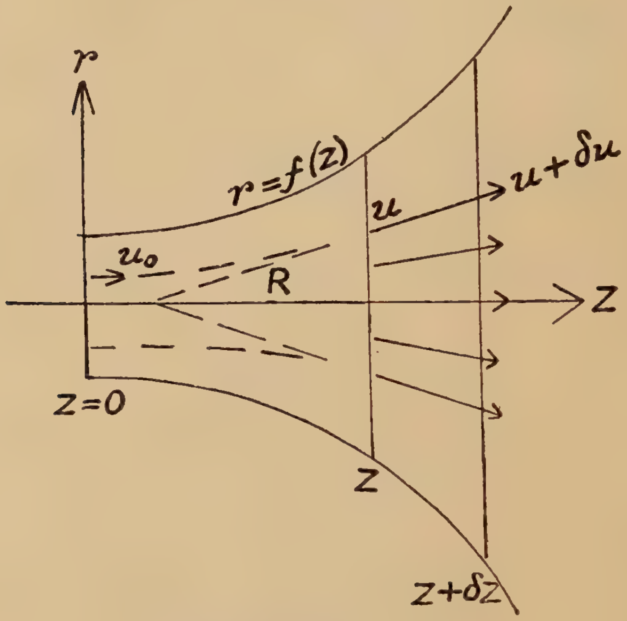
$$\frac{dr}{dz}=\delta=\left(\frac{r}{r_0}\right)\delta_0=\frac{r}{f(z)}\cdot\frac{df}{dz}=r\cdot\frac{d}{dz}(\ln f(z)). \quad \ldots \quad (4.1)$$

We immediately obtain the stream lines as the family represented by the equation

$$r = \mu f(z), \qquad \qquad (4.2)$$

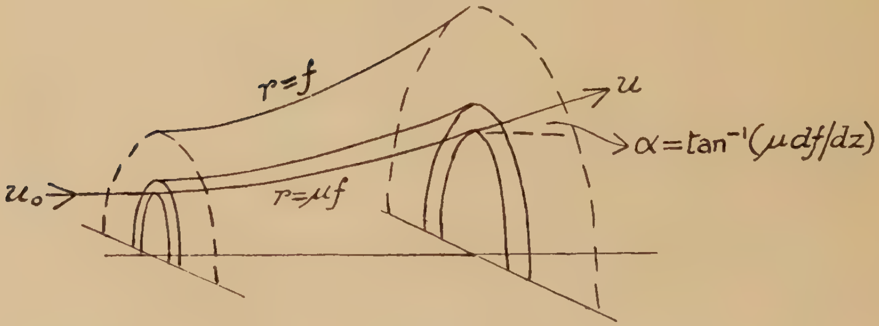
where μ may be any constant between 0 and 1.

Fig. 3.



Velocity distribution in a nozzle.

Fig. 4.



We now consider the motion of the fluid contained in the elementary volume bounded by the stream surfaces $r=\mu f(z)$ and $r=(\mu+d\mu)f(z)$ (see fig. 4). The equation of continuity is now sufficient to determine the velocity u at any point. The flux across the annulet at $z=0$ is $2\pi\{\mu f(0)\}\{d\mu f(0)\}\rho_0 u_0$ which must equal the final value of this quantity, $2\pi\{\mu f(z)\}\{d\mu f(z)\}\rho u \cos\alpha$. We therefore obtain

$$\left(\frac{u_0}{u}\right)\left\{\frac{f(0)}{f(z)}\right\}^2=\frac{\rho}{\rho_0}\cos\alpha=\left(\frac{Q^2-u^2}{Q^2-u_0^2}\right)^{\frac{1}{\gamma-1}}\left\{1+\mu^2\left(\frac{df}{dz}\right)^2\right\}^{-\frac{1}{2}} . \quad . \quad (4.3)$$

In cases of practical interest, the factor $\{1+\mu^2(df/dz)^2\}^{-\frac{1}{2}}$ may be considered to be unity, and the working equation for u assumes the form

$$\left(\frac{u_0}{u}\right)\left(\frac{Q^2-u_0^2}{Q^2-u^2}\right)^{\frac{1}{\gamma-1}}=\left\{\frac{f(z)}{f(0)}\right\}^2 . \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

As $f(z)\rightarrow\infty$, it is immediately seen that $u\rightarrow Q$. It is noted that the increase in velocity is determined not by the curvature of the nozzle, but only by $f(z)/f(0)$. Thus a short tube with great curvature produces the same increase in velocity as a long tube with small curvature, if the ratios of outlet to inlet radii are equal in the two cases.

It is easy to show that if a divergent flow is being considered $\{[f(z)/f(0)]>1\}$, equation (4.4) has two roots, one lying between $u=0$ and $u=u_0$ and one lying between $u=u_0$ and $u=Q$. In the supersonic case, u is an increasing function of R and the second solution is the relevant one.

Exactly the same procedure may be followed to give the subsonic velocity of a fluid in a divergent flow. Here the velocity ultimately decreases to zero and the root of (4.4) lying between $u=0$ and $u=u_0$ is the relevant one in this case.

§ 5. Divergent Flows under Isothermal Conditions.

When the equation of state is of the form $p=a_0^2\rho$, the local speed of sound is always a_0, and Bernouilli's equation is of the form (2.4). No limiting value of the velocity exists as in the adiabatic case. Proceeding as before, we obtain, in the same notation,

$$\left\{\frac{f(z)}{f(0)}\right\}^2=\left(\frac{u_0}{u}\right)\exp\left\{\frac{1}{2a_0^2}(u^2-u_0^2)\right\} . \quad . \quad . \quad . \quad . \quad (5.1)$$

In terms of Mach number,

$$\left\{\frac{f(z)}{f(0)}\right\}^2=\frac{M_0}{M}\exp\left\{\tfrac{1}{2}(M^2-M_0^2)\right\} . \quad . \quad . \quad . \quad . \quad . \quad (5.2)$$

§ 6. Numerical Results.

For purposes of illustration it is now proposed to calculate the velocity u at the point at which some representative stream line intersects the particular section of the nozzle of radius $f(z)$ in terms of the inlet velocity on that stream line at the section of radius $f(0)$. The form of the streamline may be estimated from the considerations of § 1. If the streamline intersects the inlet section at a distance $sf(0)$ from the axis the equation of the streamline is, according to our assumptions, $r=sf(z)/f(0)$.

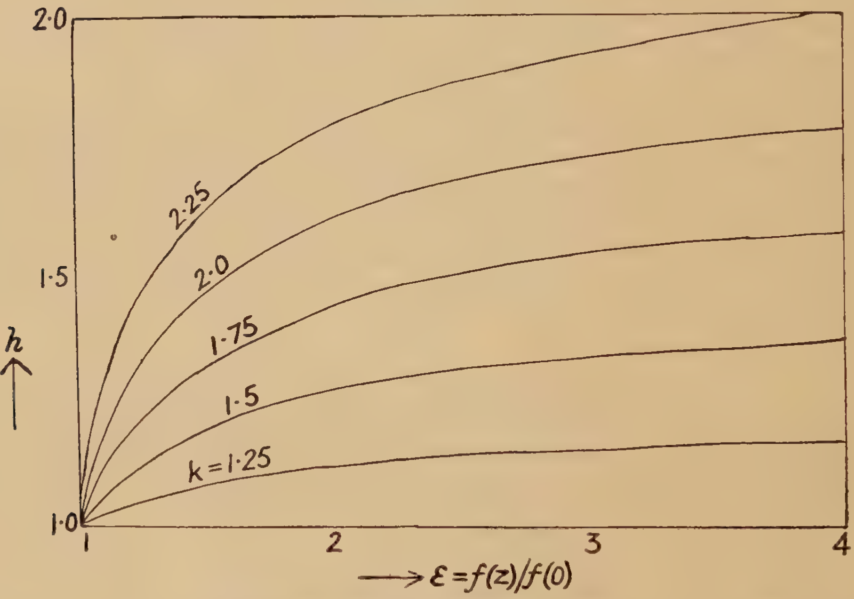
We suppose that the inlet velocity on the chosen streamline is u_0, the velocity on the stream line at the point whose abscissa is z is hu_0, and Q, which is an absolute constant for the whole field of flow, is ku_0. In supersonic flows, k is always less than $(\gamma+1/\gamma-1)^{\frac{1}{2}}$, whilst for subsonic flows, it is always greater than this value. The inlet local Mach number is obtained from Bernouilli's equation and is

$$\left(\frac{2}{\gamma-1}\,\frac{h^2}{k^2-h^2}\right)^{\frac{1}{2}}.$$

The numerical work which now arises is the tabulation of a form of equation (4.4)

$$\frac{1}{h}\left(\frac{k^2-1}{k^2-h^2}\right)^{1/\gamma-1}=\left\{\frac{f(z)}{f(0)}\right\}^2=\epsilon^2, \quad \ldots\ldots \quad (6.1)$$

Fig. 5.



where ϵ may be interpreted as the expansion ratio of the nozzle with reference to section z. If we suppose that the initial velocity u_0 and initial Mach number M_0 are known, then $Q=ku_0$ is calculated as $\{1+[2/(\gamma-1)M_0^2]\}^{\frac{1}{2}}u_0$. We may thus find the value of $f(z)/f(0)$ from equation (6.1) which will produce a velocity on that streamline of $u=hu_0$. When this velocity is known, the local Mach number is calculated from Bernouilli's equation.

The following tables give $\epsilon=f(z)/f(0)$ as a function of k and h, assuming $\gamma=1{\cdot}4$. For purpose of tabulation, it is much simpler to tabulate this quantity than to give a set of solutions of the equation for h when k and ϵ are known. From the results of Table I. a graph has been prepared giving values of $u=hu_0$ as a function of the two variables $Q=ku_0$ and $\epsilon=f(z)/f(0)$ (fig. 5).

The tables are subdivided into four sections involving subsonic and supersonic flows under adiabatic and isothermal conditions. As example of their use :

For simplicity let us assume the velocity to be constant across sections at right angles to the axis. (If this is not so, we have merely to build up the velocity field from considerations of velocities on particular streamlines.) If $Q=1{\cdot}5u_0$, under adiabatic conditions, the initial Mach number (denoted in brackets in Table I.) is 2. In a nozzle of expansion ratio 2·393, a velocity $1{\cdot}3u_0$ is attained at the exit, where the Mach number is 3·886. Table I *a*. refers to a similar flow under isothermal conditions. An inlet velocity u_0 at Mach number 2 is increased to an exit velocity $1{\cdot}3u_0$ in a nozzle of expansion ratio 1·788. The Mach number at the exit is in this case 2·6.

Thus a comparison is made between isothermal and adiabatic flows at the same inlet Mach numbers.

TABLE I.

Tabulation of $f(z)/f(0)=\epsilon$ (supersonic, adiabatic). Mach numbers are denoted in brackets. Results are given to four significant figures.

$h=\frac{u}{u_0}$ \ $k=\frac{Q}{u_0}$	1·25	1·5	1·75	2·0	2·25
1·0	1·0	1·0	1·0	1·0	1·0
	(2·981)	(2·000)	(1·557)	(1·291)	(1·109)
1·1	1·709	1·200	1·090	1·044	1·019
	(4·141)	(2·412)	(1·807)	(1·472)	(1·253)
1·2	6·112	1·570	1·232	1·113	1·053
	(7·652)	(2·982)	(2·106)	(1·677)	(1·409)
1·3	—	2·393	1·459	1·216	1·107
		(3·886)	(2·480)	(1·921)	(1·582)
1·4	—	5·249	1·849	1·369	1·184
		(5·813)	(2·982)	(2·192)	(1·777)
1·5	—	∞	2·615	1·602	1·289
		(∞)	(3·719)	(2·535)	(2·000)
1·6	—	—	4·613	1·978	1·448
			(5·044)	(2·982)	(2·261)
1·7	—	—	17·40	2·659	1·677
			(9·141)	(3·609)	(2·579)
1·8	—	—	—	4·145	2·030
				(4·617)	(2·982)
1·9	—	—	—	9·292	2·622
				(6·990)	(3·621)
2·0	—	—	—	∞	3·779
				(∞)	(4·337)
2·1	—	—	—	—	6·781
					(5·812)
2·2	—	—	—	—	25·47
					(10·42)

TABLE II.

Tabulation of ϵ (subsonic, adiabatic). Mach numbers given to three significant figures.

$h=\frac{u}{u_0}$ \ $k=\frac{Q}{u_0}$	3·0	3·25	3·5	3·75	4·0
1·0	1·0 (0·791)	1·0 (0·723)	1·0 (0·667)	1·0 (0·620)	1·0 (0·577)
0·8	1·058 (0·629)	1·068 (0·568)	1·074 (0·525)	1·080 (0·488)	1·085 (0·456)
0·6	1·173 (0·456)	1·191 (0·420)	1·204 (0·389)	1·215 (0·362)	1·226 (0·339)
0·4	1·396 (0·301)	1·423 (0·277)	1·445 (0·257)	1·462 (0·247)	1·477 (0·225)
0·2	1·941 (0·149)	1·985 (0·132)	2·018 (0·128)	2·046 (0·119)	2·069 (0·112)
0·0	∞ (0)	∞ (0)	∞ (0)	∞ (0)	∞ (0)

TABLE I *a*.

Tabulation of ϵ (supersonic, isothermal, with same initial Mach numbers as in adiabatic case).

$h=\frac{u}{u_0}$ ↓ 1·0	1·0	1·0 (2·0)	1·0 (1·557)	1·0 (1·291)	1·0 (1·109)
1·1	1·520 (2·981)	1·176 (2·2)	1·082 (1·713)	1·041 (1·420)	1·017 (1·220)
1·2	2·427 (3·279)	1·417 (2·4)	1·192 (1·868)	1·096 (1·549)	1·045 (1·331)
1·3	4·066 (3·577)	1·788 (2·6)	1·332 (2·024)	1·169 (1·678)	1·084 (1·442)
1·4	7·134 (3·875)	2·208 (2·8)	1·513 (2·180)	1·261 (1·807)	1·136 (1·553)
1·5	13·12 (4·173)	2·849 (3·0)	1·742 (2·336)	1·374 (1·937)	1·199 (1·664)
1·6	25·30 (4·472)	3·761 (3·2)	2·031 (2·491)	1·514 (2·066)	1·277 (1·775)
1 7	51·07 (4·770)	5·077 (3·4)	2 413 (2·647)	1·686 (2·195)	1·373 (1·885)
1·8	108·1 (5·068)	7·001 (3·6)	2·898 (2·803)	1·895 (2·324)	1·486 (1·996)
1·9	239·3 (5·366)	9·865 (3·8)	3·530 (2·958)	2·151 (2·453)	1·620 (2·107)
2·0	554·9 (5·664)	14·20 (4·0)	4·359 (3·114)	2·467 (2·582)	1·780 (2·218)
2·1	1347·0 (5·962)	20·88 (4·2)	5·455 (3·270)	2·856 (2·711)	1·972 (2·329)
2·2	3420·0 (6·260)	31·37 (4·4)	6 916 (3·425)	3·338 (2·840)	2·199 (2·440)

TABLE II *a*.

Tabulation of ϵ (subsonic, isothermal with same initial Mach numbers).

$h=\frac{u}{u_0}$ ↓	1·0	1·0 (0·791)	1·0 (0·723)	1·0 (0·667)	1·0 (0·620)	1·0 (0·577)
	0·8	1·056 (0·633)	1·066 (0·579)	1·073 (0·533)	1·079 (0·495)	1·084 (0·462)
	0·6	1·168 (0·474)	1·188 (0·434)	1·202 (0·400)	1·214 (0·371)	1·224 (0·346)
	0·4	1·378 (0·316)	1·417 (0·289)	1·440 (0·267)	1·459 (0·248)	1·474 (0·231)
	0·2	1·924 (0·158)	1·972 (0·145)	2·010 (0·133)	2·039 (0·124)	2·064 (0·116)
	0·0	∞ (0)	∞ (0)	∞ (0)	∞ (0)	∞ (0)

§7. PROPAGATION OF CERTAIN VELOCITY PROFILES IN NOZZLES.

A problem of interest to design engineers is the investigation of a non-uniform velocity profile in a divergent nozzle. This problem cannot properly be dealt with by one-dimensional theory as the variation of velocity across any section may be considerable. The solution may be obtained, however, from the Tables I. and II., and fig. 5. A particular problem is given below to illustrate the method.

Let us suppose that at the inlet of the nozzle, radius R_1, we have a velocity distribution with magnitudes (1, 0·98, 0·95, 0·87, 0·75)u_0 at the points with radial coordinates (0, 0·25, 0·5, 0·75, 1)R_1, where u_0 is some given velocity. In addition we suppose the flow to be supersonic, with a Q-value of 1·5u_0. The inlet distribution is thus characterized by values of k equal to (1·5, 1·53, 1·58, 1·73, 2) at the quarter radii. For the purposes of the problem, we require the velocity distributions across the sections of radii R_2 and R_3 where the expansion ratio is 1·5 and 2 respectively. These may be found as follows.

At 0·5R_1 across the inlet, $k=1{\cdot}58$. By interpolation from fig. 5, we find that if $k=1{\cdot}58$, and $\epsilon=2$, $h=1{\cdot}41$. Thus the velocity at 0·5R_3, where the expansion ratio is 2, is $(0{\cdot}95u_0\times1{\cdot}41)$, or 1·23$u_0$. By following this procedure we may estimate the velocities at the quarter radii of the exit sections in terms of u_0. For large expansion ratios, we should expect the velocity distribution to become uniform and equal to $Q=1{\cdot}5u_0$. The following results are obtained, to two places of decimals :

$\epsilon=1{\cdot}0$: (1, 0·98, 0·95, 0·87, 0·75)u_0 at (0, 0·25, 0·5, 0·75, 1)R_1,

$\epsilon=1{\cdot}5$: (1.18, 1·18, 1·16, 1·13, 1·10)u_0 at (0, 0·25, 0·5, 0·75, 1)R_2,

$\epsilon=2{\cdot}0$: (1·25, 1·25, 1·24, 1·23, 1·20)u_0 at (0, 0·25, 0·5, 0·75, 1)R_3.

§8. COMPARISON WITH ONE-DIMENSIONAL THEORY.

We may first recapitulate the results of one-dimensional approximate theory relating to flows in nozzles. Whatever the shape of the nozzle, one-dimensional theory assumes the velocity to be constant across sections perpendicular to the axis and that this velocity is wholly axial in direction. This immediately affords the result that

$$\{f(z)\}^2\rho q=\text{constant}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7.1)$$

or, using Bernouilli's equation,

$$\{f(z)\}^2q(Q^2-q^2)^{1/\gamma-1}=\text{constant}, \quad . \quad . \quad . \quad . \quad . \qquad (7.2)$$

a form which is identical to that of (4.4). Thus, when the velocity is constant across sections of constant z, the theories agree with the proviso that the velocity calculated from one-dimensional theory acts at a slope $\delta=(r/r_0)\delta_0$ where δ_0 is the slope of the nozzle at that section.

In addition, however, the theory outlined in this paper affords values for the fluid velocity and Mach number in flows in which the variation of velocity across a section is considerable. Such a case cannot be solved by the one-dimensional approach.

§9. COMPARISON WITH EXPERIMENT.

It is desirable that comparison should be made of the results derived in this paper with the results of practical experiment. Application was made to the National Physical Laboratory and to the Royal Aircraft Establishment, Farnborough, but unfortunately, very little experimental evidence is available for comparison.

It would also be interesting to compare the results given with those obtained from more exact treatment by the method of characteristics and it is hoped to carry out such an investigation at a future date.

ACKNOWLEDGMENTS.

The author is indebted to Professor L. Rosenhead for valuable criticism in the preparation of this paper. During this period the author was in receipt of a grant from the Department of Scientific and Industrial Research.

REFERENCES.

DURAND, W. F., 1934 (Ed.), *Aerodynamic Theory*, **3**, p. 230.

GOURZHIENKO, Z. A., 1939, Report No. 462, Central Aero-Hydrodynamical Institute, Moscow, reprinted in *N.A.C.A.*, T.M. 1137.

XXXIII. *Noise in Current-carrying Ohmic Conductors.*

By BERNARD MELTZER*.

[Received December 19, 1949.]

ABSTRACT.

The mean-square noise current in a conductor which is in temperature equilibrium and carries a current I, is calculated on the assumption that the current is due to elementary carriers of charge e. The value obtained for the spectral density is $\{(2k\mathrm{T}/\mathrm{R})+[2k\mathrm{T}/\mathrm{R})^2+(2e\mathrm{I})^2]^{\frac{1}{2}}\}$, where R, T, k are respectively resistance, absolute temperature and Boltzmann's constant. The interpretation of this result is that, over and above the ordinary thermal Nyquist noise, there is present an additional noise, of which the spectral density varies in square-law fashion according to $(e^2\mathrm{R}/k\mathrm{T})\mathrm{I}^2$ for small I, and in linear fashion according to $\{(-2k\mathrm{T}/\mathrm{R})+2e\mathrm{I}\}$ for large I. These predictions are compared with Bernamont's experimental results for thin metallic conductors, and the discrepancies are discussed.

INTRODUCTION.

IN a recent paper (Meltzer 1949) the ordinary thermal noise in a conductor, sometimes called Johnson or Nyquist noise, was successfully interpreted as pure shot noise. This was done with the aid of an expression, derived there,

$$\mathrm{I}_0=(k\mathrm{T}/\mathrm{R}e), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for the average current carried, in either direction, across any cross-section of a conductor in thermal equilibrium by the randomly moving elementary charge carriers. Here k is Boltzmann's constant, while T and R are the temperature and resistance of the conductor, which has its terminals short-circuited ; and it is assumed, for simplicity's sake, that all the carriers have the same effective charge e. (It was shown that the result is not changed if the latter assumption is waived.) Schottky's law for pure shot noise, applied to $2\mathrm{I}_0$, the total average random current, immediately gives the correct value for the Nyquist noise.

The analysis was carried out for the case when there is no source of power in the circuit, so that the nett current is zero. Below is presented the generalization of the analysis to the case when the conductor carries a steady current. The result obtained is that in this case, over and above the Nyquist noise, there is present additional noise which, for small steady currents, varies as the square of the latter, and for large values tends asymptotically to a linear law of variation with steady current.

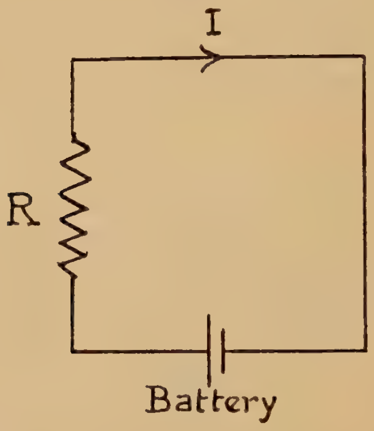
---

[*] Communicated by the Author.

This appears to be precisely one of the main features of the extra noise observed in thin metallic conductors (Bernamont 1934a, 1934b, 1937), and the square-law variation seems to be characteristic also of the noise observed in semi-conductors, according to Campbell and Chipman (1949). The other chief feature of this additional noise, its variation with frequency, is not predicted by the theory in its present state of development, although a suggestion is given below of how a fuller theory may account for it. A detailed quantitative comparison with Bernamont's experimental results, discussed below, shows that there are some suggestive and encouraging agreements with the theory as to order of magnitude, but by no means perfect overall agreement. The discrepancies are discussed.

ANALYSIS OF FLUCTUATIONS IN A STEADY CURRENT IN A RESISTANCE.

Fig. 1.



Let the steady current=I.

If Q is the nett amount of charge carried across any cross-section of the conductor in time t, the mean square value of Q^2 is given by a simple generalization of Einstein's result, quoted in the author's previous paper :

$$\overline{Q^2}=(2kT/R)t+I^2t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let us analyse Q, assuming that the elementary carriers all carry a charge e. We have

$$Q=e_1+e_2+\dots+e_i+\dots+e_n, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where n is the number of elementary carriers that cross the cross-section in time t, and

$$e_i=\pm e,$$

the positive or negative sign depending on whether the particular carrier concerned was going, say, from left to right, or from right to left.

Let p=chance that $e_i=+e$, and q=chance that $e_i=-e$, so that

$$I=(p-q)(en/t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, from equation (2),

$$Q^2 = \Sigma e_i^2 + \sum_{i \neq j}\Sigma e_i e_j .$$

Hence

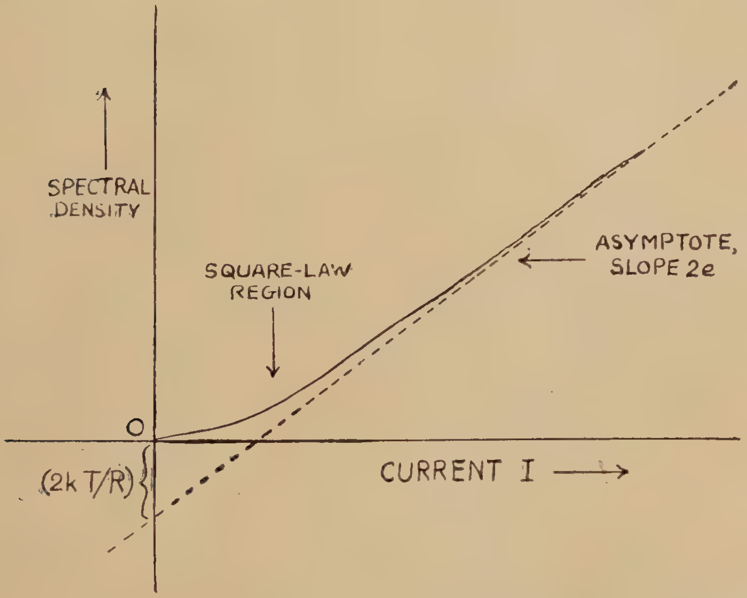
$$\overline{Q^2} = \Sigma \overline{e_i^2} + \sum_{i \neq j}\Sigma \overline{e_i e_j}$$

$$= ne^2 + n(n-1)(p-q)^2 e^2 . \qquad (5)$$

Equating the right-hand sides of equations (2) and (5), gives

$$(ne/t) = (2kT/Re) + I^2(ne/t)^{-1} . \qquad (6)$$

Fig. 2.



Now (ne/t) is the gross amount of charge carried per unit time across the cross-section by all the carriers, those going from left to right and those going from right to left. Hence to obtain the noise produced in a given frequency band, f to $f+\Delta f$, we must apply Schottky's formula ($2e\Delta f$ times current) to this current. Solving equation (6) gives

$$(ne/t) = (kT/Re) + [(kT/Re)^2 + I^2]^{\frac{1}{2}} .$$

But we have seen that the amount of current $(2kT/Re)$ accounts for Nyquist noise, hence the additional noise is associated with the current

$$-(kT/Re) + [(kT/Re)^2 + I^2]^{\frac{1}{2}} .$$

The application of Schottky's formula gives the mean-square additional noise current

$$\overline{(i_f)^2} = \{-(2kT/R) + [(2kT/R)^2 + (2eI)^2]^{\frac{1}{2}}\}\Delta f . \qquad (7)$$

The graph of spectral density of this noise is shown in fig. 2.

At first, for small I, the spectral density follows a square law given by

$$\overline{(i_f)^2}/\Delta f=(e^2\mathrm{R}/k\mathrm{T})\mathrm{I}^2\,; \quad \ldots \ldots \quad (8)$$

and then, for large I, it approaches a linear, but not proportional, law given by

$$\overline{(i_f)^2}/\Delta f=-(2k\mathrm{T}/\mathrm{R})+2e\mathrm{I}. \quad \ldots \ldots \quad (9)$$

COMPARISON WITH EXPERIMENTAL RESULTS.

Bernamont's first two papers (1934a, 1934b), although historically important, may be ignored for our purposes, since—according to his third paper (1937)—the results are vitiated by a failure to take into account appreciable changes of resistance in the platinum wire used.

In the third paper he dealt with three specimens listed below :

A : Tungsten (" Loewe " resistance) ; 518,000 ohms.
B : Tungsten (" Loewe " resistance) ; 190,000 ohms.
C : Platinum (" platine blanc ") ; 31,200 ohms.

The results we shall consider are for the total additional mean-square noise voltage appearing across the terminals of the specimen over an effective frequency band $\Delta f=1200$ cycles per second, close to the zero end of the frequency scale. The ranges of steady current used and mean-square noise voltage observed for the three specimens are as follows :

A : 0·3 to 137 microamperes ; 28 to $2{\cdot}7\times10^5$ (microvolts)^2.
B : 48 to 490 microamperes ; 9 to 830 (microvolt)^2.
C : 10 to 750 microamperes ; 0·9 to 2700 (microvolt)^2.

If mean-square voltage $\overline{(v_f)^2}$ is plotted against steady current I on linear scales, the curves for all three specimens have the general shape of the graph of fig. 2, except that the one for B does not extend into the approximately linear region. The asymptotic slopes of the other two are given by the following :

A : $2{\cdot}36\times10^{14}$ e.m.u.
C : $4{\cdot}75\times10^{11}$ e.m.u.

Now according to equation (9) of the theory above this slope should be given by

$$2e\mathrm{R}^2\Delta f.$$

Hence we may calculate effective values of the elementary charge e :

A : $e=0{\cdot}37\times10^{-20}$ e.m.u.
C : $e=20\times10^{-20}$ e.m.u.

These are to be compared with the electronic charge of $1{\cdot}6\times10^{-20}$ e.m.u.

If the results be re-plotted on logarithmic scales, one can obtain the law of variation over the initial non-linear portions of the curve :

$$\text{A}: \ \overline{(v_f)^2}=2{\cdot}88\times10^{20}\text{I}^2 \text{ e.m.u.}$$
$$\text{B}: \ \overline{(v_f)^2}=3{\cdot}80\times10^{15}\text{I}^2 \text{ e.m.u.}$$
$$\text{C}: \ \overline{(v_f)^2}=7{\cdot}94\times10^{15}\text{I}^2 \text{ e.m.u.}$$

According to equation (8), the law should be

$$\overline{(v_f)^2}=(e^2\text{R}^3\Delta f/k\text{T})\text{I}^2 ;$$

hence effective values of e may again be calculated (assuming $\text{T}=290$ degrees Kelvin) :

$$\text{A}: \ e=0{\cdot}83\times10^{-20} \text{ e.m.u.}$$
$$\text{B}: \ e=0{\cdot}014\times10^{-20} \text{ e.m.u.}$$
$$\text{C}: \ e=0{\cdot}30\times10^{-20} \text{ e.m.u.}$$

It is seen that for A and C there are discrepancies respectively of factors of 2 and 67 in the values of elementary charge effective over the square-law and linear regions, respectively, of the observations. On the other hand, all the values, except the single one for B, are within an order of magnitude of the electronic charge. [Another test of the theory which is less encouraging, however, is the following : On the linear scale graphs of A and C, the asymptotes should according to equation (9) cut the current axis at $\text{I}=(k\text{T}/\text{R}e)$, so that another estimate of effective e may be made from the values of the intercepts. This gives :

$$\text{A}: \ e=3{\cdot}4\times10^{-23} \text{ e.m.u.}$$
$$\text{C}: \ e=6{\cdot}8\times10^{-23} \text{ e.m.u.]}$$

It is therefore clear that the analysis given above is inadequate as a complete quantitative explanation of Bernamont's results. Furthermore it does not predict any frequency dependence of the noise, which is an experimentally determined fact. In fact it seems probable that it is just this defect of the theory that is responsible for a large part of the discrepancies calculated above. For, some measurements carried out by Bernamont on the platinum specimen C showed a drop in average spectral density by a factor of 1000 for a 1000-fold increase in frequency ; since the effective pass band of the experiments considered above cover as much as 1200 cycles per second at the lower end of the frequency scale, it is clear that it would have been more appropriate to use results over a much smaller frequency band to check the theory.

A possible explanation of the frequency dependence has been suggested by Dr. R. Fürth in discussion. The effect might be similar to Debye and Falkenhagen's account (1928) of the frequency dependence of the conductivity of electrolytes. For the effective charge of an elementary particle which crosses a given cross-section may be not merely its own charge e but a value modified by the existence of a charged "atmosphere" around it. In electrolytes, such an atmosphere is created because the Coulomb forces exerted, say, by a given positive ion produce in its neighbourhood an excess of negative ions and a defect of positive ones, on the average ; for semi-conductors one might expect something similar, electrons taking the place of negative ions and holes the place of positive ions ; in the case of metals, the main effect is presumably a defect of electrons in the neighbourhood of a given electron. If the relaxation time associated with the production of such an atmosphere during the movement of the particle or hole concerned, is not very much smaller than the period of the frequency, one would expect frequency-sensitive effects. On the face of it, such effects should produce a lowering of effective charge, and it is therefore perhaps significant that of the seven values of charge calculated above, all but one are smaller than the charge of an electron.

The discrepancies described above may also, possibly, be associated with experimental error, or the assumption of only one kind of effective charge. The theory can be re-stated for a number of different types of charge carriers, just as was done for the case $I=0$ in the author's previous paper, but does not then lend itself to comparison with the rather sparse experimental data publicly available at present.

The author wishes to thank Dr. R. Fürth for stimulating and helpful advice.

References.

Bernamont, J., 1934 a, *Compt. Rend.* (Paris), **198**, 1755 ; 1934 b, *Ibid.*, **198**, 2144 ; 1937, *Ann. de Phys.*, Ser 11, **7**, 71.
Campbell, R. H., and Chipman, R. A., 1949, *Proc. I.R.E.*, **37**, 938.
Debye, P., and Falkenhagen, H., 1928, *Phys. Zs.*, **29**, 121.
Einstein, A., 1906, *Ann. Phys. Lpz.*, **19**, 371.
Meltzer, B., 1949, *Phil. Mag.*, **40**, 1224.

XXXIV.—*A 4-dimensional Generalization of Wilson's Hypothesis.*

By A. Papapetrou,
I.C.I. Research Fellow, University of Manchester *.

[Received January 31, 1950.]

Summary.

Wilson's hypothesis of electric currents accompanying the rotation of electrically neutral matter (in order to explain the origin of the geomagnetic field) is generalized in a 4-dimensional language. The main consequences of the generalized formulation are discussed.

Several attempts at formulating a unified theory of gravitation and electromagnetism have been made, but none of them has led to satisfactory results. In spite of this failure, it seems to be almost certain that, if a correlation of the two fields really has a physical meaning, it must, first of all, explain the origin of the magnetic fields of the earth, the sun and other stars in the way recently proposed by Blackett (1949). It therefore might be useful to start the discussion from the other end, and try to see whether the demand of an explanation of the geomagnetic field could give some hints about the nature of the dependence of the electromagnetic on the gravitational field. A simple type of correlation of the two fields suggested by such a discussion will briefly be described in the following.

From a phenomenological point of view the geomagnetic field is sufficiently described by Wilson's hypothesis (Wilson 1923). This postulates that a rotating material ring will carry the current density

$$\mathbf{s} = -\frac{1}{c}\sqrt{(\mathrm{G})}\rho\mathbf{v}, \quad \dots \dots \dots \quad (1)$$

ρ being the mass density, $\mathbf{v}$ the peripheral velocity of rotation, and G the gravitational constant. Hence it will be sufficient to demand from the new postulate, which we are going to formulate, that it must reduce to Wilson's formula (Wilson 1923) when applied to a rotating ring. A second demand which we have to impose is that there must be no contradiction of the general principles of Maxwell's theory. In its simplest form the last demand is equivalent to the condition that the additional " gravitational " sources of the electromagnetic field, which we are going to introduce, must be describable by a charge-current 4-vector density fulfilling the continuity equation

$$\frac{\partial s^i}{\partial x^i} = 0. \quad \dots \dots \dots \quad (2)$$

---

* Communicated by Professor L. Rosenfeld.

The main difficulty of Wilson's hypothesis is that it makes use of the *three*-dimensional velocity **v**. The relativistic point of view demands a 4-dimensional formulation, and a straightforward generalization of Wilson's hypothesis for four dimensions would lead to an additional and extremely strong electric field which does not exist in nature. A simple way out of this difficulty is to correlate any given element of mass with a corresponding *central point*, and then to introduce the *difference* of the 4-dimensional velocities of the mass element and the corresponding central point into the formula defining the gravitational sources of the electromagnetic field. Such a central point is immediately suggested in Newton's theory of gravitation when we consider the gravitational field of the earth. The equipotential surfaces $\phi=$const. are almost spheres and there is in the interior of the earth one equipotential surface which shrinks to a point. We shall call this point the *gravitational centre* of the earth. There is a large number of such centres in space. The moon, the planets, their satellites if they are not too small, the sun, the stars, each of them has its own gravitational centre. Any line of force of the gravitational field starts from *one* of these centres and goes to infinity without meeting any other centre *. Thus an arbitrary mass element is, at a given time, correlated to one gravitational centre; in order to find this centre it is sufficient to follow the gravitational line of force passing through the actual position of the mass element in the direction of the gravitational force acting on the mass element. It is clear that this definition of the gravitational centre will be valid in the 4-dimensional treatment too, provided that there exists a *scalar* function corresponding to the Newtonian potential of gravitation. This possibility exists even in Einstein's theory of gravitation (general relativity) when interpreted as a theory of gravitation in a flat space (Rosen 1940, Papapetrou 1948). Besides g_{ik}, which is then only the potential of gravitation, we have the metrical tensor η_{ik} of flat space, and there is the scalar

$$\phi=\eta^{ik}(g_{ik}-\eta_{ik})$$

which differs very little from the Newtonian potential of gravitation. (There seems to be no possibility of defining the gravitational centre in the usual form of general relativity working with Riemannian space only.)

Having defined the gravitational centre corresponding to a given mass element at a given time, we can now immediately formulate our postulate. We call u^i the 4-dimensional velocity of the mass element:

$$u^i=\left\{\frac{\mathbf{v}}{c\sqrt{(1-\beta^2)}};\ \frac{1}{\sqrt{(1-\beta^2)}}\right\},\qquad \beta=\frac{v}{c}, \qquad . \quad . \quad . \quad (3)$$

---

* Because there are only positive masses. The picture is comparable with that of an electrostatic field arising from charges of one sign only.

and U^i the velocity of the corresponding gravitational centre *. Then we put

$$p^i = \sqrt{(G)}\rho(u^i - U^i), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and introduce the quantity ω fulfilling the equation

$$\Box \omega = \frac{\partial p^i}{\partial x^i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Finally we define the gravitational charge-current density by the relation

$$s_i = \frac{\partial \omega}{\partial x^i} - p_i, \qquad (p_i = \eta_{ik} p^k). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

One verifies immediately that the vector s_i thus defined fulfills the continuity equation (2).

It remains to be proved that in the case of the earth our postulate leads to the magnetic field calculated from Wilson's formula (1). We first notice that for all mass elements inside and near the surface of the earth the gravitational centre is a point situated very near to the mass centre (or the geometrical centre) of the earth †. Thus if we use the frame in which the centre of mass is at rest, we shall have

$$U^i = (0, 0, 0, 1).$$

We see that, since $\beta = v/c \ll 1$, then

$$u^4 = 1/\sqrt{(1-\beta^2)} \approx 1$$

and consequently according to (4) $p^4 \approx 0$ and

$$p^i = \{\mathbf{p};\ 0\}, \qquad \mathbf{p} = \frac{1}{c}\sqrt{(G)}\rho\mathbf{v},$$

v being the velocity of the mass element relative the the mass centre of the earth. For a uniform ring around the earth's axis we shall then have

$$\frac{\partial p^i}{\partial x^i} \approx \operatorname{div} \mathbf{p} = 0,$$

and consequently from (5)

$$\omega = 0.$$

---

* It is easy to see that the position of the gravitational centre will be almost simultaneous with the position of the mass element. The gravitational field of the planets is nearly static (because of their velocities being small compared with c) and consequently the lines of force—*i. e.* the trajectories of the hyper-surfaces ϕ=const—will almost be situated on the hyperplanes t=const.

† The main displacement of the gravitational centre of the earth relative to its mass centre is caused by the sun and it amounts to less than 5 km. For the creation of a new gravitational centre near the surface of the earth by an accumulation of mass of the earth's density a body of dimensions not much smaller than those of the earth would be necessary.

It follows from (6)

$$\mathbf{s} = -\frac{1}{c}\sqrt{(G)}\rho\mathbf{v}, \qquad s^4 = 0,$$

i. e. in first approximation we have Wilson's current distribution (1) with no charge distribution present.

From the experimental point of view the case of an isolated body near the surface of the earth will be more interesting than that of a circular ring, since one might hope to be able to detect and measure in the laboratory the magnetic field produced by such a body. We therefore calculate the magnetic field produced by an isolated body according to postulate (6). Since the velocities will be small compared with c, we omit all time derivatives, and find from (5)

$$\Delta\omega = -\operatorname{div}\mathbf{p}.$$

This is the same equation as that giving the potential of a dipole distribution and its solution is

$$\omega = -\frac{1}{4\pi}\int\frac{1}{r^3}(\mathbf{pr})\,dv.$$

We then see from (6) that the lines of current have the geometrical form of the lines of forces of a dipole distribution whose density is proportional to **p**. From the current density **s** we have to calculate the magnetic potential **A** by means of the equation

$$\Delta\mathbf{A} = -4\pi\mathbf{s},$$

and then the magnetic field **H** = curl **A**. Therefore

$$\Delta\mathbf{H} = -4\pi\operatorname{curl}\mathbf{s} = 4\pi\operatorname{curl}\mathbf{p}.$$

The integration of this equation gives

$$\mathbf{H} = -\int\frac{1}{r}\operatorname{curl}\mathbf{p}\,dv,$$

which after a partial integration leads to the formula

$$\mathbf{H} = -\int\frac{1}{r^3}[\mathbf{pr}]\,dv. \qquad \dots\dots\dots (7)$$

We see that the magnetic field produced by the mass element contained in the volume element dv is identical with that given by Biot–Savart's formula for a current element $-\mathbf{p}\,dv$.

Since **p** is linear in **v**, it follows that if the body performs a complicated motion which can be reduced to a number of simpler ones, the total magnetic field produced by the body will be the sum of the magnetic fields corresponding to these simpler motions*. The following two examples give characteristic consequences of our postulate (6). A projectile moving relative to its terrestrial surrounding with the velocity **V**, while the peripheral velocity of rotation of the earth at the place of the experiment is $\mathbf{v}_0$, will produce the magnetic field corresponding to the velocity $\mathbf{v}=\mathbf{V}+\mathbf{v}_0$. (If, *e. g.* $V=3v_0$ and both V znd v_0 have the same direction, the moving projectile will produce a field four times stronger than if it were at rest.) A rotor rotating in the laboratory will produce a magnetic field which is the sum of the following two fields: a Biot–Savart field corresponding to the translation of the rotor's mass with the velocity $\mathbf{v}_0$, and a dipole field corresponding to the rotation of the rotor around its axis (the magnetic moment of this dipole being given by Blackett's formula). Since in any experiment we cannot obtain velocities much larger than v_0, we see that the magnetic field will always be very small. These consequences are essentially different from those following from the generalization of Wilson's hypothesis suggested by Blackett (1949a): According to the latter there will be no general superposition of the magnetic fields corresponding to different motions of a body, and there is the possibility of the production of much stronger fields either by suitable motions or by suitable arrangement of other masses around the test body.

---

* Strictly speaking the motion of a body in the laboratory will influence the position of the gravitational centre of the earth, *i. e.* the values of U^i. But this will be vanishingly small because of the smallness of the mass of the test body relative to the total mass of the earth.

More complicated consequences follow from the postulate (6) for certain astronomical cases in which large displacements of the gravitational centre occur. For a qualitative discussion it is sufficient to consider a planet of radius R and a satellite of radius r at the distance a from the planet. Assuming for simplicity that both bodies have the same density, we find by an elementary calculation that the satellite will have its own gravitational centre *only if the distance a is larger than a critical value* a_0 given by the relation

$$(a_0-r)^2=\frac{R^3}{r}.$$

Thus a satellite, moving on an elliptic orbit such that the distance a takes values larger *and* smaller than a_0, will have its own gravitational centre only on the outer part of its orbit (*i. e.* the part for which $a>a_0$). Hence on the outer part of the orbit the satellite will produce the magnetic field corresponding to its own rotation (around its own gravitational centre). On the inner part ($a<a_0$) the gravitational centre of the satellite will coincide with that of the planet; therefore the satellite will also produce the magnetic field corresponding to its translational velocity round the planet*.

Similar considerations will hold for a double star, though in this case further complications will arise if the masses of the two stars are approximately equal (and if the two stars are not fully separated).

---

* This again contrasts with the corresponding consequence of the generalization of Wilson's hypothesis suggested by Blackett (1949 a), according to which a body moving freely in a gravitational field will only produce the magnetic field corresponding to its own rotation.

Anyway, the general qualitative conclusion from the postulate (6) is that *large periodic variations of the magnetic field of a double star are to be expected, if the orbits are strongly elliptic.* In this connection it is interesting to recall that Babcock (1948) has observed a star whose magnetic field is periodically changing. But very little is known at present about the internal structure of this star, so that it is difficult to see how this observation could give an argument for or against the postulate (6).

As a final remark we ought to stress that the postulate formulated above does not constitute a unified field theory. It might give a successful description of the one physical feature of a unified theory, namely the production of an electromagnetic field by the gravitational field of moving (neutral) bodies. Further we must stress that the formulation is not purely differential. It involves the position of the gravitational centre of the earth (or generally of the gravitational field surrounding the test body), which can only be determined by solving the equations of the gravitational field. This is a most unusual feature, but we do not think that there are *a priori* arguments against it. It is very difficult to see how one would have to proceed in order to set up a complete theory using the postulate (6) ; but some way might be found, if only there were some direct experimental evidence in favour of the postulate.

REFERENCES.

Babcock, H. W., 1948, *Phys. Rev.*, **74,** 489.

Blackett, P. M. S., 1947, *Nature*, **159,** 658 ; 1949, *Phil. Mag.*, **40,** 125 ; 1949 a, Discussion at R. Astr. Soc. reported in Observatory, **69,** 100.

Rosen, N., 1940, *Phys. Rev.*, **57,** 147.

Papapetrou, A., 1948, *Proc. Roy. I.* A, **52,** 11.

Wilson, H. A., 1923, *Proc. Roy. Soc.* A, **104,** 451.

XXXV. CORRESPONDENCE

Further Evidence for the Existence of τ-Mesons.

By J. B. Harding,
Imperial College, London *.

[Received February 15, 1950.]

We have recently observed an event in Ilford G5 200 μ "Nuclear Research" Emulsions, and this we have to interpret in terms of the spontaneous decay of a τ-meson, *i. e.* a particle of mass of approximately 1000 m (where m is the electronic mass).

A facsimile drawing of the event is shown. The star centre is 6 mm. from the edge of the plate. The angle from the upward vertical to track K, which has a length of 740 μ from the glass to the star centre, is approximately 85° at the emulsion-glass interface. In fig. 2 we have plotted the grain density (number of grains per 70 μ) v. range from the star centre of tracks a, b, and K, and show for comparison the variation of the mean grain density with *residual range* of thirty protons and ten μ-mesons. Unfortunately, track c passes out of the emulsion/air surface after 30 μ and has a grain density of $29 \pm 8{\cdot}3$ (the error being the standard deviation on the number of grains counted).

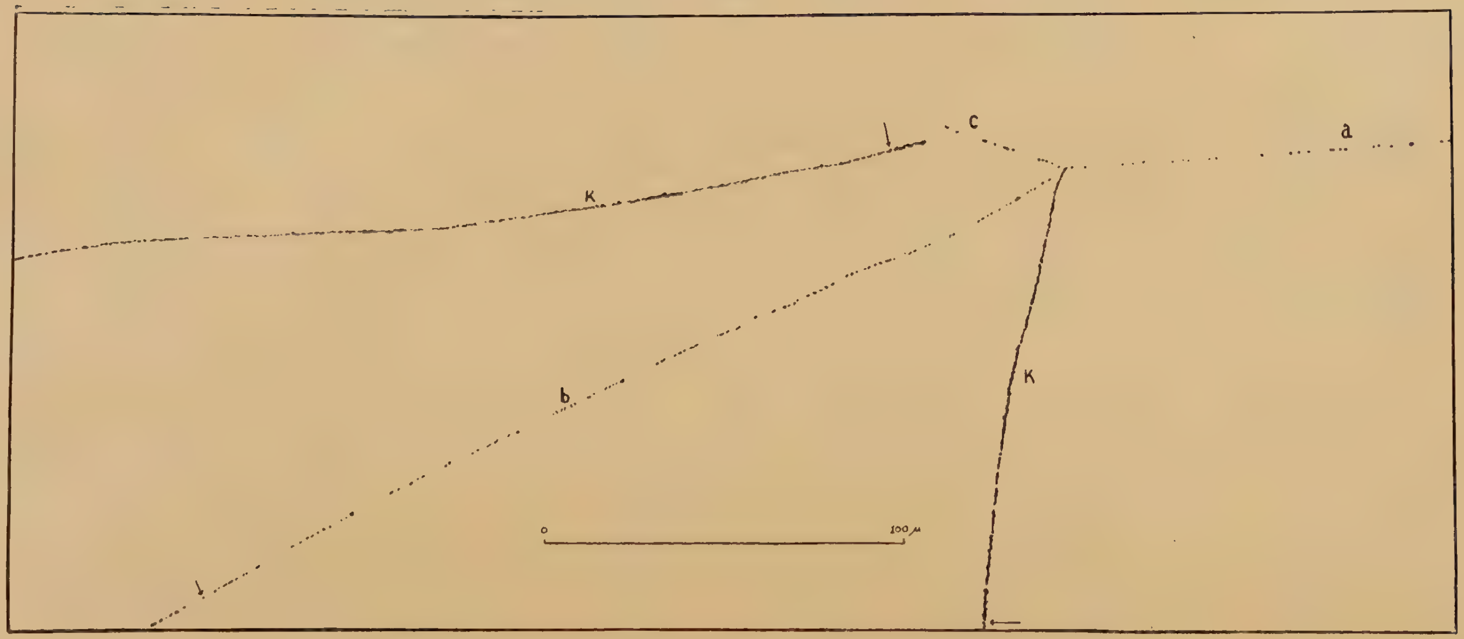
Track b, although it showed considerable multiple scattering, remained almost in the plane of the emulsion for 1300 μ before undergoing a large angle single scatter. It was therefore possible to determine the product momentum × velocity ($\rho\beta$) of the particle which produced this track, by measuring this multiple scattering. The mean projected angle of the track was measured at intervals of 60 μ, and the difference in angle, α, between neighbouring cells thus obtained. The mean of twenty readings of α gave $\bar{\alpha} = 1{\cdot}0° \pm 0{\cdot}15°$ and hence we are 99 per cent confident that $16 < \rho\beta < 46$ MeV., the most probable value being 25 MeV. The scattering of track a (range from star to glass $= 420$ μ) was too small for measurement, and we are thus confident that the spurious scattering is much less than 1/10° per 60 μ. The frequent multiple scattering of track K which increased towards the star centre was more characteristic of a typical π-meson than of a proton.

After allowing for the shrinkage of the emulsion during processing (for details see Harding, Lattimore, and Perkins 1949) it was found that the normal to a plane defined by any two of the tracks, a, b, c, was perpendicular to the third, within 2°, *i. e.* these three tracks were coplanar. Track K did not lie in the same plane.

---

* Communicated by Sir George Thomson.

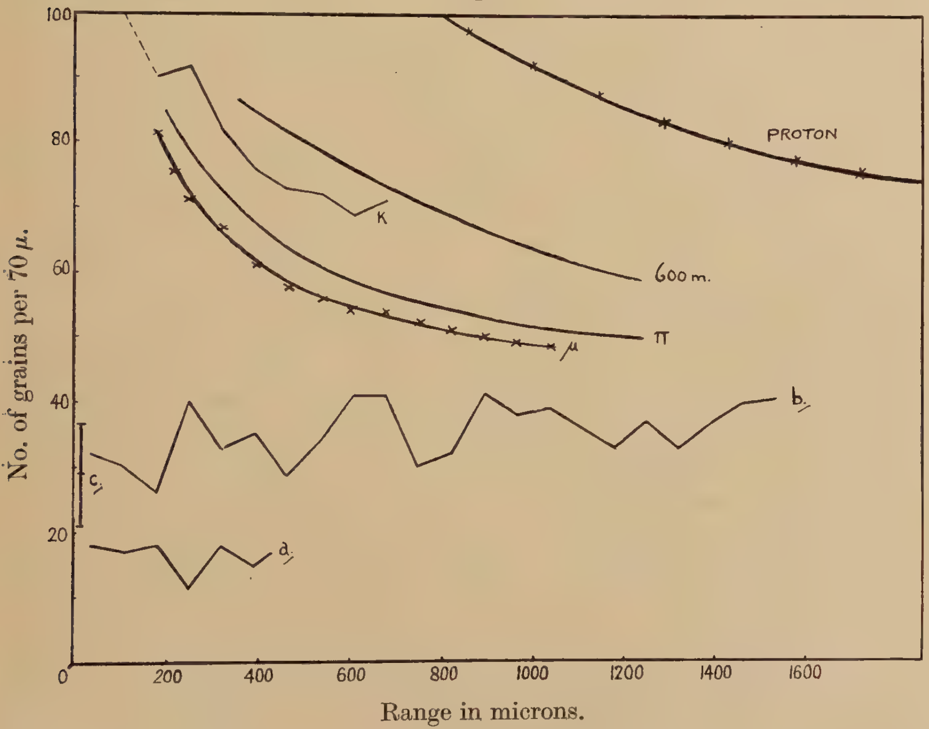
Fig. 1.



Facsimile drawing made with the aid of a projection microscope.
The two sections of track K meet at the arrows.

Before interpreting these experimental results we must first discuss the recording characteristics of the emulsion which was exposed for 85 days under three metres of ice. In view of the long exposure, there is a certain amount of fading, which, coupled with the low development, makes the recording of tracks at minimum ionization uncertain. We do not feel justified in comparing grain densities of tracks formed in unrelated events, to obtain quantitative information about any particular track. Rather, we must analyse this event on its own merits, and merely include data from other events in order to show that the fading problem cannot alter our conclusions. We have obtained grain density measurements on proton and μ-meson tracks that stop in the emulsion and obtain no significant difference in the grain-densities of those coming from the glass and those from the surface. Therefore the development (temperature method) of the plate is uniform.

Fig. 2.



Both the scattering and grain density of track K decrease with increasing distance from the star centre. We thus conclude that particle K is travelling *towards* the centre of the star and presumably produces it. The grain density measurements indicate that particle K must stop, or at least come very nearly to rest before producing the star. The only generally accepted particles which so behave are π-mesons which give up their rest mass to form stars with excitiation energies of less than 150 MeV. The expected grain density-range curve for π-mesons (as constructed from the curve for

μ-mesons) is also indicated in fig. 2. If, therefore, particle K is a π-meson, this event must have occurred towards the end of the exposure, for the fading must be extremely low as the grain-density of this track is markedly above the average grain density that we would expect for π-mesons. Assuming tracks a, b and c to be formed by protons, we find energies of >300, 110–140, 100–300 MeV. respectively. Thus the total minimum energy of 510 MeV. could not be supplied by the rest mass of a π-meson. We cannot exclude the possibility that track a was formed by a fast electron, but even then its energy must have been greater than 50 MeV. as the scattering was too small to measure. Allowing an improbable fluctuation in grain density, track c could possibly be produced by a slow energy electron. In this way we could obtain a minimum excitation energy of 160 MeV. As yet, no evidence for the production of fast electrons following the nuclear capture of a π-meson has been reported. Also it seems improbable that track b could be formed by a proton of 110 MeV. which would have $p\beta=175$ MeV. ($\bar{\alpha}=0{\cdot}15°$), the observed value is only 25 MeV. This track cannot be produced by an electron of 25 MeV. for it should then have minimum grain density, which must be $\leqslant 16$, since we have observed a track of such a grain density in this event.

It seems probable that a meson must have produced track b. There is clearly insufficient energy available for a π-meson to produce a μ-(216 m.) meson (for this is the only lighter one known to exist) and to supply kinetic energy to particle a. In no way then can this event be interpreted along orthodox lines of the nuclear capture or of the spontaneous decay of a π-meson.

The fact that the tracks a, b and c are coplanar is very strong evidence for the spontaneous decay of a particle at rest. Such a process, namely the decay of a τ-meson, of mass $\sim 1000\,m$., into a π^- and probably two other π-mesons, has already been published by the Bristol group (Brown *et al.* 1949). If we assume K represents the track of a τ-meson, the grain density measurements would now show that the latent image fading is by no means negligible and hence energies that we infer for particles a, b and c (assumed now to be π-mesons) cannot be very accurate. It would seem, however, that their energies should lie in the intervals >50, 10–20, 5–10 MeV. respectively. For $p\beta=25$ MeV., a π-meson has kinetic energy 13 MeV. Thus for particle b, there is good agreement between the energies estimated from grain counting and scattering measurements. Resolving moments first perpendicular and then parallel to track a, we find that the energies of a and c are 50 and 22 MeV. respectively. Allowing 450 MeV. for the rest energies of the three π-mesons, we find the rest energy of the τ-meson to be 535 MeV. ($\equiv 1040\,m$.).

If we assume the decay products to be μ-mesons, particle b will again have 13 MeV. kinetic energy (for $p\beta=25$ MeV.) and the energies of a and c are 49 and 21 MeV. respectively. These are still in agreement with the energies (now >38, 7–15, 4–30 MeV. for a, b and c) that we infer from grain counting. Allowing 330 MeV. for the rest energy of the three mesons, the τ-meson would have a rest energy of 413 MeV. ($\equiv 810\,m$.).

It is seen, therefore, that there is good internal consistency if we interpret this event as the decay at rest of a τ-meson of mass about $800\,m.$ or $1000\,m.$, depending on the decay scheme. Since this event is explained in exactly the same way as was the event of Brown *et al.* (1949), the existence of τ-mesons would seem to be truly confirmed and as such should be duly considered in any meson theory of nuclear forces. From the theoretical point of view, the range of nuclear forces might well correspond to an admixture of π- and τ-mesons. Since the particles are so rarely observed (see for example, Franzinetti 1950, in which negative evidence for the existence of τ-mesons was obtained), they must be rarely produced or have a short lifetime. By now, of the order of 10,000 lightly ionizing particles produced in nuclear disintegration must have been observed in photographic emulsions and yet no evidence for the decay in flight of a τ-meson has been obtained. Assuming that τ-mesons are produced with kinetic energies about three times their rest mass, we conclude that their proper lifetime cannot be much smaller than 10^{-9} secs. if they are to account for a large fraction of these fast particles. With such a lifetime we would then expect to observe more coming to rest and it therefore seems that the hypothesis that they are only rarely produced is necessary for agreement with experimental facts.

I wish to express my appreciation to Professor Sir George Thomson for a helpful and interesting discussion on this event.

Note added in proof.—One further example of a particle coming to rest in the emulsion and producing three nearly coplaner lightly ionising particles, has just been observed. If the incoming particle were assumed to be a π-meson, the fading must be negligible and the three ejected particles, if protons, would have a total kinetic energy of >600 MeV. Two of the tracks could be formed by fast electrons and only in this case would the energy be low enough to be supplied by the rest mass of a π^{-}-meson. The event can easily be explained as the spontaneous decay of a τ-meson into three lighter mesons with total kinetic energy >70 MeV. or >95 MeV. if the decay particles are assumed to be μ- or π-mesons respectively. Slight distortion, and the shortness of the tracks, makes a detailed study of this event impossible.

Department of Physics,
Imperial College, London.

J. B. Harding.

References.

Brown, Camerini, Fowler, Muirhead, Powell, and Ritson, 1949, *Nature*, **163**, 82.
Franzinetti, 1950, *Phil. Mag.*, **41**, 86.
Harding, *et al.*, 1949, *Proc. Roy. Soc.* A, **196**, 325.

Intrinsic Block Structure and the Strength of Metals.

By M. S. Paterson, Ph.D.,
Aeronautical Research Laboratories, Melbourne*.

[Received February 6, 1950.]

In his recent paper "On the Strength of Quasi-Isotropic Solids", Dr. Fürth (1949) discusses the idea of an "intrinsic block structure" of vital importance in the processes of melting and rupture. He also suggests that this block structure may be identical with that assumed in Bragg's theory of the strength of metals and the explanation by Wood and co-workers of the X-ray line broadening in cold-worked metals. However, from the proposed form of the dependence of the block size on temperature (increasing with decreasing temperature), Fürth's hypothesis would predict a decrease of X-ray line broadening and strength with decrease of temperature. This is contrary to experiment, for it has been observed that the X-ray line broadening and the stress–strain curve for metals are higher at lower temperatures (Wilson and Thomassen 1934, Hutchison 1948, Paterson and Orowan 1948, Paterson 1950). It is therefore doubtful whether the mechanical properties of a metal are influenced by an intrinsic block structure such as that proposed by Fürth.

It should also be made clear that Bragg's theory deals with the resistance of a metal to plastic flow, whereas Fürth's deals with its rupture. In the case of specimens that "neck" during a tensile test, the "ultimate tensile strength" is a measure of the resistance to plastic flow at a certain point on the stress–strain curve (see, for example, Orowan 1949). Rupture occurs at a later stage in the test, under stresses that cannot be determined from the experiment owing to the complex stress state then existing in the necked region of the specimen.

References.

Fürth, R., 1949, *Phil. Mag.*, [7], **40**, 1227.
Hutchison, T. S., 1948, *Nature*, **162**, 374.
Orowan, E., 1949, *Rep. Prog. Phys.* (Phys. Soc.), **12**, 185.
Paterson, M. S., and Orowan, E., 1948, *Nature*, **162**, 991.
Paterson, M. S., 1950, to be published.
Wilson, J. E., and Thomassen, L., 1934, *Trans. Amer. Soc. Metals*, **22**, 769.

---

* Communicated by the Author.

XXXVI. *Notices of New Books and Periodicals received.*

Theoretical Hydrodynamics. 2nd Edition. By L. M. MILNE-THOMSON. (Macmillan & Co. [600 pp.] Price £3.

THE first edition of this work was published in 1938 ; in producing the second edition the author has maintained the vector presentation of the subject. For the most part, the book is concerned with the theory of incompressible inviscid flow ; there is, however, a chapter on viscosity incorporating a section on boundary layers, as in the former edition, and in addition in the second edition a chapter on compressible flow, subsonic and supersonic.

The presentation of much of the two-dimensional irrotational flow has been modified in the second edition by the use of the " circle-theorem " which though primarily concerned with the effects of introducing a circular cylinder in a given flow has, through conformal transformation, repercussions on aerofoil problems. The corresponding " sphere-theorem " in three dimensional flow is also introduced. The treatment, in the reviewer's opinion, benefits considerably by way of systematization by these changes.

For many problems of inviscid flow vector notation can be used with advantage as here, but particularly in such topics as rate-of-strain and stress in a viscous fluid its limitations are, in the reviewer's opinion, only too obvious.

L. H.

Photoelasticity. By H. T. JESSOP and F. C. HARRIS. [Pp. 184.] (London : Cleaver-Hume Press). Price 28*s*.

THIS is a timely volume on the principles and methods of photoelasticity suitable for the engineer. Since it is written essentially for the experimenter the reader only requires an elementary knowledge of mathematics and no previous knowledge at all of optical polarization on which adequate chapters are included. There are chapters on reduction and interpretation of results and on materials and procedure. The book ends with examples of the use of the method and with some appendices of a rather more mathematical nature.

The Extrapolation, Interpolation, and Smoothing of Stationary Time Series with Engineering Applications. By NORBERT WIENER. [Pp. 163]. (New York : Technology Press of M.I.T. and John Wiley & Sons, Inc., 1949). Price $4·00

THIS important and long-awaited monograph is " an attempt to unite the theory and practice of two vitally important fields of work—time series in statistics, and communication engineering ". It mainly consists of a basic mathematical theory of linear prediction, linear wave filters, and related problems like approximate differentiation in the presence of " noise ", developed by Wiener during the war and first published in a restricted report. The present publication includes also a useful introductory account of the theoretical concepts required, and as appendices two reprinted articles by N. Levinson giving a more elementary mathematical treatment, and a numerical table of Laguerre functions for use with an orthogonal series solution of the prediction problem.

Engineers may find Wiener's discussion rather too abstract for immediate application, though approximations directed towards the representations of ideal solutions by realizable electrical networks are indicated. Mathematicians and statisticians will welcome a much more comprehensive account than that available in the same author's *Cybernetics*, even if the development is in places still somewhat elliptical ; one problem, at least in many statistical fields of application, that of first determining precisely the required correlation functions or associated spectra, is not seriously discussed.

M. S. B.

The Physics of Rubber Elasticity. By L. R. G. Treloar. [Pp. vii+255, with 113 text figures.] (Oxford: Clarendon Press 1949.) Price 21*s*.

This is one of the series of Monographs on the Physics and Chemistry of Materials published by the Oxford University Press. It is designed to give an account of recent experimental and theoretical work on the elastic properties of rubber, especially the large elastic extensibility of natural and synthetic rubbers and the type of molecular structure with which it is associated.

The book opens with an account of the general physical properties of rubber and the changes in the thermodynamical functions when a sample of rubber is deformed. This thermodynamical analysis provides evidence of the reversible nature of the deformation of rubbers, and it gives a firm foundation for the statistical developments which follow. These are expounded in the succeeding chapters which deal with the elasticity of long chain molecules and of molecular networks. Different types of strain are then examined, and it is shown that there is good agreement between the experimental results and the statistical theory in the region of small strain, but deviations in the region of very large strain. This topic is considered further in the closing chapter in which an account is given of some of Rivlin's work in this field. There are also chapters on photo-elastic properties, crystallization and mechanical properties, and relaxation phenomena .

The author's remarks following his equation (2·1) are open to objection. He claims that this equation is a definition of internal energy : this claim implies acceptance of the atavistic notion of " heat " which is made untenable by the experiments on which the first law of thermodynamics is based. The experimental basis of the first law is provided by the experiments which Joule described in his memoir of 1849 to the Royal Society. These experiments were done under *adiabatic* conditions. They indicate the *existence* of a function of state which is called the internal energy and the changes in which are equal to the work done on the system under adiabatic conditions. The author's equation (2·1), which refers to *non-adiabatic* processes, can then be used to define " quantity of heat ". It is also a pity that the author has not followed the recommendations of the International Union of Chemistry (1947) and of Physics (1948) in his notation for the thermodynamical functions.

These blemishes do not, however, detract from the very readable account of the properties of natural and synthetic rubbers given in this monograph.

A. R. Miller.

Twelve Lectures on Theoretical Rheology. By Markus Reiner, [Pp. 163.] (North Holland Publishing Company.) Price 9 Dutch florins.

Professor Reiner's well-known " Ten Lectures " have been revised and expanded into the present twelve. The new edition is much more clearly and accurately printed than the old. The main change in content is the adoption of a rigorous tensor notation for the components of strain, with the use of the Kronecker delta to separate the isotropic and deviator components, leading to some simplifications and generalization of the formulæ. The book is an excellent brief account of the mathematical basis of rheology. Further discussions of the connection between " solid " and " liquid " viscosities and of the dimensional basis of " consistency variables " would have been useful, and the argument at the top of page 133 seems to me unconvincing.

F. R. N. Nabarro.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]